VOLUME 38 JUNE, 1952 NUMBER 6

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Publication Office: Mack Printing Company, Easton, Pa.
Editorial Office and Home Office of the Academy
2101 Constitution Avenue, Washington 25, D. C.

Entered as Second-Class matter, June 21, 1930, at the Post-office at Easton, Pennsylvania, under the Act of August 24, 1912. Accepted for mailing at a special rate of postage provided for in section 1103, act of October 3, 1917. Authorized June 18, 1920.

Published Monthly Price per Annum, $7.50

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PROCEEDINGS
OF THE
NATIONAL ACADEMY OF SCIENCES

Volume 38 June 15, 1952 Number 6

EXPERIMENTS ON THE EFFECTS IN HETEROZYGOUS CONDITION OF SECOND CHROMOSOMES FROM NATURAL POPULATIONS OF DROSOPHILA WILLISTONI

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Communicated by Th. Dobzhansky, March 31, 1952

The Problem.—Natural populations of all species of Drosophila so far studied in this respect carry a great amount of concealed genetic variability, consisting of recessive mutant genes and chromosomal aberrations of the inversion type. Many of the mutant genes are lethal or semilethal in homozygous condition.[1] The effects of these lethals on heterozygous flies are insufficiently known, and conflicting views about these effects have been expressed by several authors. Berg[2] showed that the lethals found in natural populations of *D. melanogaster* are deleterious in heterozygous condition, and Stern and Novitski[3] found the same to be true for lethals which arise in the laboratory. Dubinin[4] considered, however, the natural lethals in the same species to be completely recessive, but he quotes the results of Masing who regarded many of the natural lethals to be heterotic. In *D. pseudoobscura* Wright, Dobzhansky and Hovanitz[5] found that autosomal lethals are less frequent in natural populations than they would be if they were completely recessive and if the populations were wholly panmictic. The discrepancy may be due either to deleterious effects of most lethals in heterozygotes, or to local inbreeding in natural habitats, or to a combination of both causes. Muller[6] emphasized the importance of the viability effects of heterozygosis for deleterious recessive genes for an understanding of the dynamics of populations of sexual organisms, including man. He also referred to unpublished experiments on *D. melanogaster*, which indicated that mutants which are lethal when homozygous are deleterious in heterozygotes.

The experiments reported in the present article have been carried out to investigate the effects on the viability of the flies of heterozygosis for different chromosomes found in natural populations of *Drosophila willistoni*. Chromosomes which are lethal to homozygotes as well as chromosomes

which permit the homozygotes to survive (the "Normals") have been studied. It appears that combinations of these chromosomes give rise to a variety of genotypes which differ in their viabilities under the conditions of the experiments.

Material and Methods.—Samples of wild populations of *D. willistoni* were collected at Quarai, Emboaba and Caxias, in the state of Rio Grande do Sul, and at Catuni, in the state of Bahia, in Brazil. The effects in homozygous condition of second chromosomes found in these samples were studied as described in a previous publication.[7] A total of 52 second chromosomes which were lethal in homozygous condition, and 60 second chromosomes which produced homozygotes close to normal viability were selected. All the lethals were known not to be allelic.

The lethal and the "Normal" chromosomes are maintained in separate strains in balanced condition. A chromosome which contains the dominant mutant Star, the recessives brown and abbreviated, and a pericentric inversion which suppresses crossing over in heterozygotes, is used as a balancer.[7]

Four series of experiments were arranged, as shown in table 1. In the first series, flies which carried Star (and the other genes) in one second chromosome, and a lethal in the homologous chromosome, were crossed to flies homozygous for a known Normal second chromosome. The offspring of this cross should consist of Star flies and wild-type flies (heterozygous for the lethal) in approximately equal numbers. In the second series of experiments, flies which carried one Normal second, and one Star second, chromosome were crossed to flies homozygous for a different Normal second chromosome. The offspring of this cross should, theoretically, consist also of Star and of wild-type flies in equal numbers. In the third series, flies heterozygous for the Star chromosome and for different (non-allelic) lethal chromosomes were intercrossed. Since the Star chromosome is lethal when homozygous, the offspring of this cross should consist of about two-thirds Star and one-third of wild-type flies. The fourth series of experiments differed from the third in that the chromosomes opposite to Star were known not to be lethal when homozygous. The offspring should, theoretically, consist of Star and wild-type flies in a ratio approaching 2:1.

A summary of the results is given in table 1. A total of 359,667 flies were counted. They came from 735 cultures, representing 198 different combinations of second chromosomes. The cultures were kept at 25°C. on banana agar medium; the flies that hatched were counted on alternate days until the ninth day after the beginning of the eclosion of adults in a given culture.

Incomplete Recessivity of the Lethals.—The theoretical ratios, of 1 Star:1 wild-type, or 2 Star:1 wild-type, expected in the two categories of the

crosses can be realized only if the Star and the wild-type flies were equal in relative viability under the conditions which obtained in our experiments. In reality, the carriers of the Star chromosome were known to be less viable than their wild-type sibs.[7]

The ratios of Star: wild-type flies are, however, not uniform in the different crosses (table 1). It can be seen that the wild-type class is less frequent in the offspring of the Star/lethal to Normal crosses (55.20 per cent

TABLE 1

A SUMMARY OF THE EXPERIMENTAL DATA

TYPE OF CROSS	LOCALITY	COMBINATIONS OF CHROMOSOMES	NUMBER OF CULTURES	TOTAL FLIES COUNTED	PER CENT WILD IN TOTAL	MEAN FREQUENCY OF WILD-TYPE
Star/lethal ×	Quarai	32	102	54,724	60.42	60.94
Normal/Normal	Catuni	37	135	100,292	55.33	54.91
	Emboaba	20	172	76,500	51.29	55.67
	Total	89	409	231,516	55.20	
Star/Normal ×	Quarai	16	42	19,979	60.33	64.70
Normal/Normal	Catuni	2	29	14,876	56.16	56.80
	Emboaba	2	38	14,716	55.94	56.90
	Total	20	109	49,571	57.78	
Star/lethal ×	Quarai	13	21	8,251	43.87	43.85
Star/lethal	Catuni	24	49	20,712	40.26	39.25
	Emboaba	3	7	5,068	44.87	43.53
	Caxias	18	106	33,471	41.10	41.31
	Total	58	183	67,502	41.47	
Star/Normal ×	Quarai	5	6	1,610	44.35	45.46
Star/Normal	Catuni	19	19	4,760	45.29	46.73
	Emboaba	6	6	3,642	49.45	48.36
	Caxias	1	3	1,066	43.15	42.60
	Total	31	34	11,078	46.32	

in the total) than in the offspring of the Star/Normal to Normal crosses (57.78 per cent in the total). The obvious difference between these two categories of crosses is that in the first of them the wild-type flies are heterozygous for a chromosome which is known to be lethal to homozygotes. This suggests that, on the average, the lethal chromosomes used in our experiments are deleterious also in heterozygous condition. This suggestion is confirmed by comparison of the outcomes of the Star/lethal × Star/lethal crosses with those of the Star/Normal × Star/Normal crosses (table 1). The frequency of the wild-type class in the progeny of the former is lower (41.47 per cent) than in the latter (46.32 per cent). The wild-type flies in the progeny of Star/lethal × Star/lethal crosses are

heterozygous for two (non-allelic) lethals, while their Star sibs are heterozygous for only one lethal (not counting Star itself).

The second-chromosome lethals used in our experiments seem, on the average, to reduce the viability of the heterozygotes. This inference can be tested further by examining the frequencies of the wild-type class separately in various series of experiments summarized in table 1. The lethal and the Normal chromosomes were derived from four different localities—Quarai, Catuni, Emboaba and Caxias. Seven possible comparisons can thus be made of the frequencies of the wild-type class in crosses which were free of lethals, and which contained lethals. These comparisons are as follows:

	STAR × NORMAL		STAR × STAR	
	LETHAL	LETHAL-FREE	LETHAL	LETHAL-FREE
Quarai	60.42	60.33	43.87	44.35
Catuni	55.33	56.16	40.26	45.29
Emboaba	51.29	55.94	44.87	49.45
Caxias	...	...	41.10	43.15

In six out of the seven comparisons, the lethal-containing wild-type class is less numerous than the lethal-free class in the corresponding crosses.

In the comparisons just considered, the percentages of wild-type flies were computed from the total yield of the flies obtained in all the crosses of a given type. It will, however, be shown below that the outcomes of the crosses involving different combinations of the chromosomes are significantly heterogeneous, as are different cultures involving the same combination of chromosomes. Mean frequencies of wild-type flies were therefore calculated, first averaging the frequencies for the different cultures of each combination of chromosomes, and then averaging the average frequencies for the different combinations of chromosomes. Since the numbers of cultures, and of flies within the cultures, obtained for different combinations of chromosomes were unequal, these mean frequencies differ somewhat from the frequencies weighted in accordance with the numbers of the flies counted. The mean frequencies are shown in the rightmost column in table 1. Mr. Allan Birnbaum and Prof. Howard Levene, of the Department of Mathematical Statistics, Columbia University, kindly devised a method for applying the analysis of variance to the data here discussed. Table 2 shows the differences between the mean frequencies of the wild-type flies in the lethal-bearing and in the lethal-free progenies of the various crosses. The table also shows the standard deviation of these differences, from which t values are derived. All the differences are negative, i.e., the lethal-bearing wild-type classes appear to survive less often than the corresponding lethal-free classes. The probability that all the differences would have the same sign by chance is 1/64. Considered separately, only two of the seven t values are significant; however, these

are also the only two based on enough data to have adequate sensitivity.[9] Combined t values for all the Star × Normal and all the Star × Star crosses are significant, and the combined t value for all seven crosses, −3.96, is highly significant.

Estimation of the Magnitude of the Average Deleterious Effect Produced by the Lethals in Heterozygous Condition.—The difference between lethal-free and lethal-bearing classes of flies is not an adequate measure of the relative viability of these classes, since this difference depends on the frequencies of the Star and wild-type flies, and therefore on the viability of the carriers of the Star chromosome. To measure the deleterious effect of heterozygosis for the lethals, a "coefficient of inviability"[8] is used. This coefficient is not identical with the selection coefficient (S); it measures the

TABLE 2

DIFFERENCE BETWEEN THE MEAN FREQUENCIES (IN PER CENT) OF THE WILD-TYPE CLASS IN THE LETHAL-CONTAINING AND THE LETHAL-FREE CROSSES

CROSS AND LOCALITY	DIFFERENCE	STANDARD DEVIATION	
Star × Normal			
Quarai	−3.76	1.52	−2.47[a]
Catuni	−1.89	3.61	−0.52
Emboaba	−1.23	3.68	−0.33
Combined Star × Normal	...	...	−1.92[a]
Star × Star			
Quarai	−1.61	2.60	−0.62
Catuni	−7.48	1.52	−4.90[b]
Emboaba	−4.83	3.51	−1.38
Caxias	−1.29	5.10	−0.25
Combined Star × Star	...	...	−3.57[b]
Combined all crosses	...	...	−3.96[b]

[a] Significant.
[b] Highly significant.

proportion of the lethal-bearing individuals which are eliminated due to differential mortality between the egg stage and the hatching of the adult insect, in relation to the lethal-free individuals which survive.

The coefficients of inviability were calculated only for the crosses Star/lethal × Normal/Normal and Star/Normal × Normal/Normal. The other crosses, which contain two lethals, raise the problem of the possible interactions of the deleterious effects of heterozygous lethals. The coefficient proved to be 14.8 per cent for Quarai, 6.1 per cent for Catuni and 0.8 per cent for Emboaba, giving a mean estimate of 7.2 per cent.

Heterogeneity of Effects of Different Chromosomes.—The data presented above show that, on the average, individuals heterozygous for lethals have a viability lower than individuals which are free of lethals. It is, however, probable that different lethals, as well as different chromosomes free of

lethals, affect the viability to different extents. An analysis of variance has been carried out under the direction of Mr. Allan Birnbaum. Its results are summarized in table 3.

As indicated above, different lethal-bearing and lethal-free (Normal) chromosomes have been utilized in the experiments. A total of 89 different combinations of chromosomes have been made in the crosses of the type Star/lethal × Normal/Normal (table 1). Some of these combinations have, however, been replicated in several culture bottles; a total of

TABLE 3

ANALYSIS OF VARIANCE OF FREQUENCIES OF WILD-TYPE FLIES OBSERVED IN DIFFERENT CROSSES

TYPE OF CROSS	SOURCE	SUMS OF SQUARES	DEGREES OF FREEDOM	MEAN SQUARES	F
Star/lethal × Normal/Normal	Between chromosomes	13,132	86	152.7	
	Within chromosomes	11,246	320	35.1	4.345
	Total	24,378	406		
Star/Normal × Normal/Normal	Between chromosomes	1214	17	71.4	
	Within chromosomes	3010	89	33.8	2.112
	Total	4225	106		
Star/lethal × Star/lethal	Between chromosomes	1846	54	34.2	
	Within chromosomes	3461	125	27.7	1.234
	Total	5307	179		
Star/Normal × Star/Normal	Between chromosomes	1516	27	56.1	
	Within chromosomes	362	3	120.7	0.465
	Total	1878	30		

409 cultures were raised for crosses of this type. The analysis of variance shows that the variance between different combinations of chromosomes is significantly greater than that between cultures of the same combination of chromosomes. The F value equals 4.345, which is significant at a level below 0.5 per cent (table 3). Similarly, for the crosses of the type Star/-Normal × Normal/Normal, the F value is 2.112, which is significant at about the 2.5 per cent level (table 3). The crosses of the type Star/lethal × Star/lethal and Star/Normal × Star/Normal do not give significantly greater heterogeneity between combinations of chromosomes than be-

tween replicate cultures. This is probably due to the small number of replications made in this series of experiments.

This result is important. It shows that the so-called "Normal" flies in a natural population are in reality a diversified collection of genotypes having different reaction norms and different adaptive values. The chromosomes which carry recessive lethals, as well as chromosomes free of such lethals, contribute to the adaptive polymorphism of the normal species population. There is an indication in the data that different lethal-bearing chromosomes are more diversified with respect to their effects on heterozygotes than are the lethal-free chromosomes, but this point requires a more rigorous proof than can be adduced on the basis of the data so far available. There are also indications that while some lethal-bearing chromosomes are distinctly deleterious to heterozygotes, others are not deleterious, or even heterotic. It should also be noted that there exists a significant heterogeneity of the viability effects between the chromosomes derived from different localities (table 1).

Acknowledgments.—The writer wishes to express his appreciation to Mr. Allan Birnbaum, Prof. Howard Levene and Mr. Timothy Prout for advice in the statistical treatment of the data; to Prof. Th. Dobzhansky for his help in the preparation of the manuscript; to Miss F. Wolff and Mrs. Malvina Cordeiro for assistance in the execution of the experiments. A part of the work has been done during the tenure of a Fellowship provided by the Rockefeller Foundation.

Summary.—The effects in heterozygotes of lethals found in second chromosomes of natural populations of *Drosophila willistoni* have been studied. On the average, the viability of lethal heterozygotes is significantly lower than that of lethal-free individuals. The coefficient of inviability is of the order of 7 per cent. Both lethal-bearing and lethal-free chromosomes vary significantly with respect to their viability effects in combinations with other chromosomes.

[1] Dobzhansky, Th., *Genetics and the Origin of Species*, 3rd ed., New York (1951).

[2] Berg, R. L., *C. R. (Doklady) Acad. Sci. URSS*, **36**, 212–218 (1942).

[3] Stern, C., and Novitski, E., *Science*, **108**, 538–539 (1948).

[4] Dubinin, N. P., *Genetics*, **31**, 21–38 (1946).

[5] Wright, S., Dobzhansky, Th., and Hovanitz, W., *Ibid.*, **27**, 363–394 (1942).

[6] Muller, H. J., *Am. J. Human Genetics*, **2**, 111–176 (1950).

[7] Pavan, C., Cordeiro, A. R., Dobzhansky, N., Dobzhansky, Th., Malogolowkin, C., Spassky, B., and Wedel, M., *Genetics*, **36**, 13–30 (1951).

[8] Moree, Ray, *Am. Natur.*, **86**, 45–48 (1952).

[9] Footnote by Howard Levene. A preliminary Chi-square test showed significant heterogeneity between replicate cultures. Because of this the usual Chi-square tests and the formula pq/n for binomial variance are inapplicable and were never used. For all further work the percentage of wild flies per culture was treated as a normal variable with a variance to be found empirically. The analysis of variance in table 3 was an ordinary one-way analysis of variance with unequal numbers of replications, each

locality being treated separately and the sums of squares then combined. Since this analysis of variance showed significant differences between different combinations of chromosomes, for further work the average of the frequencies of wild-type flies for the different cultures of a single combination of chromosomes was considered as the basic variable. Then Star/Lethal × Normal/Normal at Emboaba, for example, might have a smaller frequency of wild type than Star/Normal × Normal/Normal at Emboaba simply because the chromosome combinations of the first type chosen by chance happened to give a smaller frequency of wild-type. We can say the difference is due to the presence of the lethal chromosome only if it is significantly larger than the variability between chromosome combinations. This can be tested by an ordinary two sample t test as in table 2. Because of small numbers in many of the fourteen groups an estimate of the error variance based on all the data was used. Then t is approximately normally distributed with unit variance, and the t values can be combined by adding them (taking account of signs) and dividing by the square root of the number of t's added. This combined value will again be approximately normal with unit variance. Returning to the example of Emboaba, Star/lethal × Normal/Normal is based on twenty chromosome combinations and has adequate accuracy. Unfortunately, Star/Normal × Normal/Normal is based on only two combinations and this is the limiting factor in the sensitivity of the comparison. Most of the labor in obtaining the twenty combinations is wasted; in fact, three combinations of each type would have given a more accurate comparison. A more delicate and complicated test, based on the analysis of variance, gave substantially the same results. Mr. Birnbaum and the author of this footnote hope to discuss the statistical aspects more fully elsewhere.

SELECTION AGAINST HETEROZYGOTES FOR AUTOSOMAL LETHALS IN NATURAL POPULATIONS OF DROSOPHILA WILLISTONI

By Timothy Prout

Columbia University, New York

Communicated by Th. Dobzhansky, March 31, 1952

In an accompanying article, Cordeiro shows that the second chromosomes of *Drosophila willistoni* which are lethal when homozygous, produce a significant lowering of the viability of heterozygous flies as well. The method used in the experiments of Cordeiro is comparison of the ratios of flies heterozygous for lethals and of flies which are free of such lethals. Cordeiro estimates that, under the environmental conditions of his experiments, the average lowering of the viability produced by heterozygosis for a second chromosome lethal is approximately 7% (his coefficient of inviability). The present note reports an attempt to compare the average adaptive values of individuals heterozygous for second chromosome lethals and of individuals free of such lethals in natural populations of *D. willistoni*.

The genetic structure of natural populations of *D. willistoni* has been

studied by Pavan, *et al.*,[1] who have published data on the frequencies of second chromosomes which are lethal or semilethal to homozygotes. Dobzhansky, Spassky and Spassky[2] have obtained estimates of the mutation rates which produce these lethals and semilethals. Ives[3] and Wallace[4] have computed the probable number of loci which mutate to lethals and semilethals in the second chromosome of *D melanogaster*. The second chromosomes of *D. willistoni* and *D. melanogaster* are homologous and probably contain the same loci.[5] Professor C. Pavan[6] has kindly communicated to us his unpublished data on the frequency of allelism of second chromosome lethals isolated from various Brazilian populations of *D. willistoni*.

The method used for the estimation of the adaptive value of heterozygotes for lethals in natural populations involves the comparison of the rates of the origin of such lethals by mutation and of their elimination in homozygotes. These rates should be equal if all lethals in natural populations were completely recessive. Wright[7, 8] has given the necessary equations which describe the relationship just stated (Wright's notation will be followed throughout). For the case of complete recessivity the balance of gains and losses at equilibrium is as follows:

$$V(1 - Q) = \frac{PQ^2(1 - Q)}{1 - PQ^2} \qquad (1)$$

$$V \simeq PQ^2 \qquad (1)'$$

where:

Q = the equilibrium frequency of lethal chromosomes,
V = the mutation rate producing lethal chromosomes,
P = frequency of allelism of lethal chromosomes, i.e., the proportion of zygotes bearing two allelic lethal chromosomes among the zygotes which carry two lethal chromosomes.

The left side of equation (1) represents the rate of gain by mutation and the right side the rate of loss through homozygosis. Q, V and P may be estimated experimentally. It should be noted, however, that the experimentally determined value of P will be an underestimate if an appreciable amount of inbreeding occurs normally in the population. If the mutation rate (V) is higher than the rate of elimination (PQ^2), as is the case in all Drosophila populations studied so far, then the amount of selection against lethal heterozygotes required to account for this difference should be as follows:

$$S = \frac{V - PQ^2(1 + V)}{Q(1 + 2V) - 2PQ^2(1 + V)} \qquad (2)$$

where:

S = the mean selection coefficient of heterozygotes for lethal chromosomes.

Equation (2) still assumes random breeding. Wright[7, 8] has considered the more general case where some elimination of lethals will be due to inbreeding as well as to semidominant deleterious effects of the lethals. There is evidence, however, that the populations of *D. willistoni* are panmictic,[6] and that their genetically effective size may be taken as infinite (unpublished data of Prof. C. Pavan).

In such populations the frequency of the allelism is approximately equal to the frequency of allelism of independently arising lethal chromosomes (P_∞ according to Wright's notation). For the second chromsome of *D. melanogaster*,[3, 4] this value is $P_\infty = 0.0026$ (± 0.00057). Assuming the second chromosomes of *D. melanogaster* and *D. willistoni* carry the same

TABLE 1

FREQUENCIES OF LETHAL CHROMOSOMES (Q) AND SELECTION COEFFICIENTS OF HETEROZYGOTES FOR LETHALS (S) IN VARIOUS POPULATIONS OF *Drosophila willistoni*

POPULATION	Q	S
Belem, Pará	0.339	0.025
Rio Branco and Rio Negro	0.541	0.015
Arce	0.463	0.017
Goyaz	0.417	0.020
Catuní, Bahãa	0.443	0.018
Pirassununga, São Paulo	0.423	0.019
Mogi, São Paulo	0.386	0.021
Vila Atlantica, São Paulo	0.392	0.021
Iguassú, Paraná	0.375	0.022
Mean	0.412	0.020

loci subject to lethal mutation[5] this figure ($P = 0.0026$) can be used in equation (2), together with the estimate of the mutation rate, $V = 0.0088$, obtained by Dobzhansky, Spassky and Spassky.[2]

In table 1 there are presented the values of S obtained for nine natural populations of *D. willistoni* in which Pavan, *et al.*[1] determined the frequencies of lethal and semilethal chromosomes (semilethals are assumed to be effectively lethal under natural conditions). Different values of P were tried out by substitution in equation (2) in order to determine how great are the differences in the estimates of S which depend on experimental errors in the determination of P. If values of P twice as high or half as high as 0.0026 are used, the resulting estimates of S are not appreciably changed.

Considering the sources of error contributing to these estimates of S, a discussion of the variations of S between populations would be meaning-

less. All the examined populations of *D. willistoni* may be roughly characterized by the value obtained using the mean Q, namely $S = 0.020$. This value is of the same order of magnitude as the estimates made for natural populations of *D. pseudosbcura*,[7,8] 0.014, 0.02, 0.007 and of *D. melanogaster*,[3,9] 0.054, 0.028, 0.016, 0.011. It therefore seems justified to state that lethals in natural populations of Drosophila are on the average slightly deleterious in the heterozygous condition.

The above estimates for natural populations vary from 0.007 to 0.05 for different species and different populations. The experiments of Cordeiro reported in the preceding article show, however, that under laboratory conditions the viability of heterozygotes for lethals in *D. willistoni* is lowered in some localities by as much as 14% (his coefficient of inviability), although in other localities it is only lowered by 0.8%. These estimates are not exactly comparable to the estimates of S calculated in this paper. In the first place, the experiment of Cordeiro measures the relative viabilities of different genotypes between the egg stage and the hatching of the adults from pupae under conditions of severe crowding in laboratory bottles. These estimates of S measure the adaptive values under natural conditions where larval competition for food may not always be a limiting factor. On the other hand, the relative fecundities, longevities, sexual activities, etc., of the carriers of different genotypes influence the value of S, while they do not affect the viability estimates, such as those offered by Cordeiro. Finally it should be noted that the estimates of S, as well as of the coefficient of inviability, describe merely the average situations. In reality, some lethals may be more deleterious in heterozygotes than others, as actually demonstrated by the data of Cordeiro.

Acknowledgments.—The writer wishes to express his appreciation for the valuable advice and suggestions of Prof. Th. Dobzhansky. This article was written during the tenure of a U. S. Atomic Energy Commission fellowship.

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[9] Dubinin, N. P., *Ibid.*, **31**, 21–38 (1946).

THE ACCUMULATION OF α-PHENYLGLYCINE BY MUTANTS OF NEUROSPORA CRASSA STIMULATED BY PHENYLALANINE AND TYROSINE*

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Communicated by J. T. Patterson, April 22, 1952

During the course of experiments testing the effects of various mutagenic agents, two mutants of *Neurospora crassa* were obtained which grew slightly on the minimal medium but required phenylalanine or tyrosine for optimal growth. Genetic tests made it evident that these strains differed from the wild type by a single gene. It is the purpose of this paper to describe these mutants, one of which accumulates large amounts of α-phenylglycine; and to present evidence in favor of the hypothesis that

TABLE 1

Growth of the Mutants with Phenylalanine or Tyrosine

Recorded mg. dry weight (average of 2) of mycelium produced in 72 hours

Substance added	UT 4	UT 39	Em 5256
Nothing	2.9	4.6	66.2
Tyrosine concentration, molar:			
0.000004	11.5	8.2	66.0
0.000020[a]	18.6	19.1	63.2
0.000040	19.9	29.7	68.2
0.000200	24.2	28.5	64.2
Phenylalanine concentration, molar:			
0.000004	1.5	12.7	64.3
0.000020	1.6	28.4	69.2
0.000200	1.3	40.8	65.0
Tyrosine and phenylalanine equimolar concentrations, molar:			
0.000004	12.0	28.4	72.8
0.000020	17.0	40.0	68.4
0.000040	22.3	46.8	76.1
0.000400	25.1	59.4	64.3

[a] The value of 0.000020 molar, frequently used in other experiments, corresponds to an addition of 0.091 mg. of tyrosine and 0.083 mg. of phenylalanine per 25 ml. of medium.

this compound may be utilized as a precursor of aromatic compounds under certain conditions. α-phenylglycine, a lower analog of phenylalanine, has heretofore not been considered a normal constituent of living matter.

Evidence has been accumulating that indicates a common precursor for phenylalanine, tyrosine and other aromatic compounds. Haskins and Mitchell[1] have described a mutant of Neurospora that may utilize cinnamic

acid as an apparent precursor for phenylalanine and tyrosine or a common intermediate. Others have postulated that quinic acid or shikimic acid may act as a precursor for the synthesis of aromatic compounds in *Escherichia coli* and some Neurospora mutants.[2,3] There are also some indications that the aromatic amino acid requirements of *Leuconostoc mesenteroides* may be relieved by phloroglucinol.[4]

Experimental.—The mutant strains, UT 39 and UT 4, were isolated as ascospores from crosses using ultraviolet-irradiated macroconidia (Em 5297a) and untreated wild type (Em 5256A) protoperithecia by the method

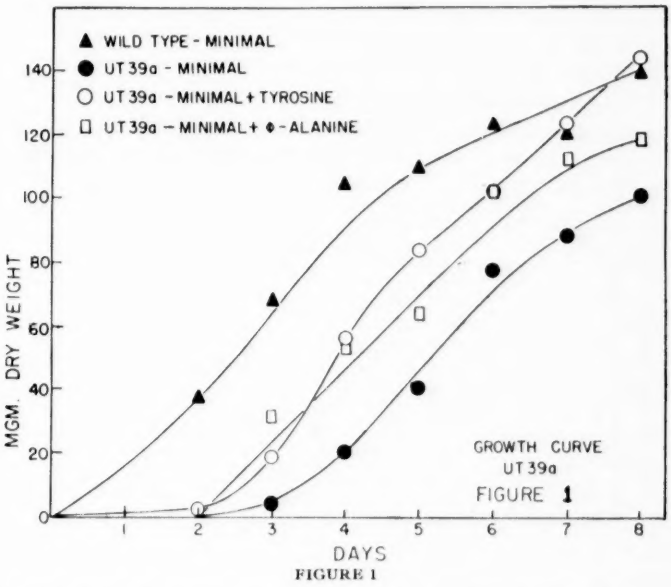

FIGURE 1

Growth of UT 39 on minimal and 2×10^{-5} *M* tyrosine and phenylalanine as compared to Em 5256A on minimal.

of Lein, Mitchell, and Houlahan.[5] UT 39 requires phenylalanine or tyrosine and UT 4 requires tyrosine for optimal growth. Growth was compared to Emerson Wild Type 5256A.

Genetic investigations by Haddox[6] showed that the mutants UT 39 and UT 4 are due to single gene differences. The genes are not allelic, UT 39 being in linkage group D of Houlahan, *et al.*,[7] and UT 4 being in another, undetermined, linkage group.

The minimal medium used in these studies was that of Beadle and Tatum.[8] The growth was measured in terms of the dry weight of mycelium produced

in 72 hours at 25°C. on 25 ml. of medium with certain exceptions as noted.

Growth Characteristics of the Mutants: UT 39 and UT 4 produce only a few milligrams of mycelium at the end of the third day of growth on minimal medium. Thereafter their growth rates increase greatly and by the eighth day they produce almost as much mycelium as wild type (Figs. 1 and 2). The addition of tyrosine or phenylalanine to the minimal medium stimulates the growth of UT 39, but not during the first two days (table 1, Fig. 1). When tyrosine and phenylalanine are present together in equimolar concentrations, however, the growth approximates that of wild

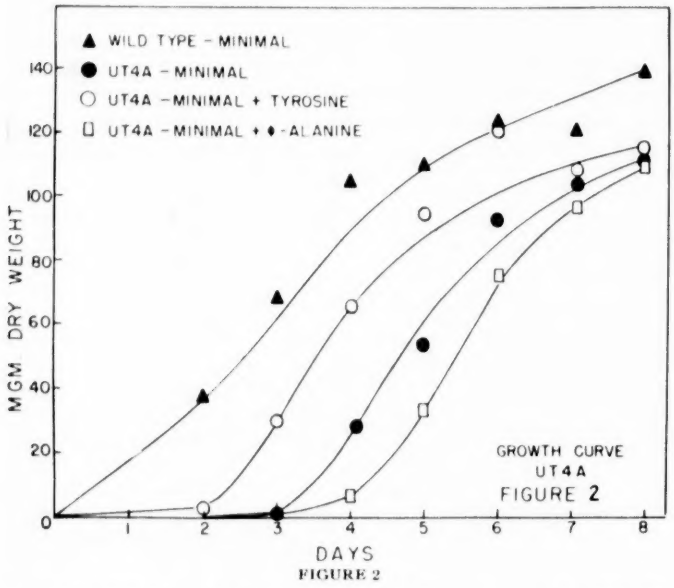

FIGURE 2

Growth of UT 4 on minimal and 2×10^{-5} *M* tyrosine and phenylalanine as compared to Em 5256A on minimal.

type after three days (table 1). UT 4 is stimulated only by phenylalanine, and the presence of both amino acids together has no significant effect on its growth rate. Neither mutant is a pH or temperature sensitive mutant.

All known amino acids, vitamins and purines and pyrimidines as well as a number of organic acids and other compounds known to have growth-promoting properties were tested to determine their effect on the mutants. In addition, iodoacetate, KCN, sulfasuxadine and taurine were tested for possible inhibition. Some of these showed an inhibitory or stimulatory

effect particularly in the presence of tyrosine or phenylalanine. These effects are summarized in table 2 in which stimulation or inhibition is given as percentage of growth of the mutant calculated on the basis of wild type growth on minimal medium as 100%. It will be noted that none of the compounds tested stimulated mutant growth as much as tyrosine or phenylalanine. Two compounds, leucine and 3-hydroxyanthranilic acid showed an unexpected inhibition. Tryptophan is known to be slightly inhibitory to wild type, and therefore its effect on the mutants is not surprising.

It was found also that iodoacetate inhibited both the mutants and wild type at concentrations of 0.001 *M*. The inhibition is partially reversed

TABLE 2

GROWTH PROMOTING OR INHIBITING SUBSTANCES

Values listed are dry weights of mutant and wild type mycelia, expressed as percentage of the average wild type mycelial dry weight when grown on minimal medium for 72 hours at 25°C.

	UT 4			UT 39			
SUBSTANCE ADDED	ALONE	PLUS TYROSINE[a]	PLUS PHENYLALANINE	ALONE	PLUS TYROSINE	PLUS PHENYLALANINE	5256, ALONE
Nothing	3.7	39.1	4.1	5.1	33.2	44.3	100.0
3-Hydroxyanthranilic acid (40 μg.)	5.1	3.8	0.0	4.4	8.7	6.5	114.0
Leucine (1.0 mg.)	2.0	8.0	3.0	3.2	10.5	6.3	98.0
Tryptophan (1.0 mg.)	2.9	3.2	3.2	2.3	25.0	31.0	75.5
α-Aminobutyric acid (0.5 mg.)	13.0	68.0	12.3	11.3	45.0	63.3	99.0
Citric acid (2.5 mg.)	10.4	78.0	5.6	6.0	66.0	79.0	108.0
Coenzyme I (0.1 mg.)	18.7	67.2	13.6	42.2	47.3	65.3	102.0
Fumaric acid (2.5 mg.)	10.2	64.8	14.2	14.2	68.0	63.7	100.0
α-Ketoglutaric acid (0.5 mg.)	14.8	63.5	16.4	32.0	59.5	58.0	100.0
Malic acid (2.5 mg.)	13.6	70.0	8.7	17.4	72.0	78.2	115.0
Succinic acid (2.5 mg.)	17.0	84.0	14.8	26.7	97.0	85.0	112.0
d-Phenylalanine (0.01 mg.)	3.0	31.0	3.2	34.0	78.0	76.0	100.0

[a] The amounts of tyrosine or phenylalanine added are those considered to be necessary for approximately half maximum growth, or 2×10^{-5} molar.

by tyrosine, phenylalanine or succinic acid (table 3) in all three strains, but whereas other amino acids are equally effective for relief of inhibition in wild type, only phenylalanine and tyrosine are effective for the mutants.

Accumulation of an Active Growth Factor: Bacterial assays of the culture media in which UT 39, UT 4 and Em 5256 had grown for a period of three days indicated an apparent accumulation of phenylalanine and tyrosine by the mutants but very little by the wild type. Using *Leuconostoc mesenteriodes* P 60, and *Lactobacillus arabinosus* USDA 17-5 after the method

of Hac, Snell, and Williams,[9] it was found that UT 39 growing on minimal medium accumulated after 72 hours 22.5 μg. per mg. dry weight of mycelium of a tyrosine active compound and 7.6 μg. of a phenylalanine active compound. Comparable figures for UT 4 were 2.5 μg. for tyrosine and 13.8 μg. for phenylalanine. Wild type produced 1 μg. or less in both assays under the same conditions.

In order to supplement the data obtained with the bacterial assays, the culture medium was chromatographed on paper. No tyrosine or phenylalanine spots could be developed with ninhydrin, but a spot did appear which in area was roughly proportional to the apparent phenylalanine and tyrosine concentrations as determined by the biological assays. The conflicting results of the bacterial assays and the paper chromatograms suggested the accumulation of some factor which satisfied the requirements of the Neurospora mutants as well as the bacteria for phenylalanine and tyrosine.

Cross feeding experiments showed that both UT 4 and UT 39 produce a compound or compounds in the medium after 3 days growth on minimal which stimulates their growth. UT 39 produces a highly active medium for both itself and UT 4, while UT 4 produces a relatively low activity compared to UT 39. Wild type medium was found to be completely inactive for either mutant, and the active factor produced by the mutants had no effect on wild type growth.

Isolation and Identification of the Active Factor: The paper chromatograms of the media from UT 39, UT 4 and Em 5256 showed a ninhydrin positive spot which had a different Rf value from the known amino acids. This substance was accumulated in large amounts in UT 39 medium only, and therefore this medium was used for its isolation.

UT 39 was grown for eight days in minimal medium, the medium filtered to free it of mycelium and concentrated twentyfold *in vacuo* at ca. 60°C. Ethanol was added to 90% by volume to precipitate salts and sugars, and the active unknown concentrated on filter paper by using the chromatopile technique of Mitchell and Haskins[10] with a water-saturated phenol solution as a solvent. The location of the active factor in the filter paper pile was determined by running supplementary paper chromatograms and using the Neurospora mutants and *Leuconostoc mesenteroides* as assay organisms. The active factor was removed from the paper with ethanol and the alcohol and residual phenol removed by distilling *in vacuo* and replaced with water. The pH of the water solution was adjusted to 7.00 with NaOH and an active crystalline precipitate was formed. This was washed with diethyl ether, 90% ethanol and water, redissolved in NaOH and the solution filtered. The solution was again returned to neutrality, filtered and the crystals washed with 95% ethanol and water. From 5500 ml. of original culture medium in which 23.32 g. of UT 39 mycelium had accumulated, 370 mg. of these purified white crystals were obtained.

TABLE 3

IODOACETATE EFFECTS

Recorded mg. dry weight of mycelia produced in 72 hours

SUBSTANCE ADDED	UT4		UT39		5256	
	ALONE	PLUS IODO-ACETATE[a]	ALONE	PLUS IODO-ACETATE	ALONE	PLUS IODO-ACETATE
Nothing	5.2	0.0	6.2	0.0	64.0	7.4
Tyrosine (0.001 molar)	24.0	10.0	31.0	15.3	58.0	20.2
Tyrosine (0.010 molar)	26.0	14.0	30.0	14.0	54.0	23.0
Phenylalanine (0.001 molar)	4.8	5.0	28.0	18.0	58.0	22.0
Phenylalanine (0.010 molar)	3.2	6.4	32.0	24.0	56.0	24.0
Succinic acid (0.001 molar)	18.2	8.2	14.8	10.0	60.0	14.0
Succinic acid (0.010 molar)	17.6	10.6	18.0	12.0	58.0	40.0
Succinic acid (0.100 molar)	20.0	18.0	22.8	17.8	60.0	48.0

[a] The concentration of the iodoacetate was 0.001 molar. The iodoacetate was sterilized by passing through a VF sintered glass filter.

TABLE 4

Rf VALUES OBTAINED FROM UNKNOWN, *dl*-α-PHENYLGLYCINE, AND CONTROLS

Average of three determinations

SUBSTANCE	SOLVENT					
	METHANOL	ETHANOL	BUTANOL-ACETIC ACID	BUTANOL-ETHANOL-AMMONIA	PHENOL-ACETIC ACID	WATER-SATURATED PHENOL
Aspartic acid	0.24	0.05	0.26	0.03	0.25	0.05
Phenylalanine	0.73	0.50	0.69	0.44	0.83	0.76
Tyrosine	0.40	0.19	0.46	0.13	0.35	0.50
Tryptophane	0.60	0.16	0.54	0.40	0.74	0.67
α-Phenylglycine	0.92	0.73	0.71	0.34	0.77	0.19
Unknown	0.92	0.75	0.71	0.33	0.76	0.19

TABLE 5

GROWTH STIMULATION BY UNKNOWN AND dl-α-PHENYLGLYCINE

Recorded mg. dry weight of the mycelia produced in three days. The Leuconostoc readings are visual estimates of proportionate growth volume

SUBSTANCE ADDED	UT4	UT39	5256	*Leuconostoc mesenteroides*
Tyrosine (0.02 mg.)	2.0	4.2	64.0	+[a]
Tyrosine (0.10 mg.)	21.2	19.4	60.0	++++
Phenylalanine (0.02 mg.)	1.8	6.2	70.0	+[b]
Phenylalanine (0.10 mg.)	2.1	21.0	64.0	++++
Unknown (0.02 mg.)	4.3	8.2	61.0	++++
Unknown (0.10 mg.)	14.8	25.2	60.0	++
dl-Phenylglycine (0.02 mg.)	5.2	7.3	57.5	++++
dl-Phenylglycine (0.10 mg.)	17.4	23.2	60.5	++
Nothing	1.0	2.0	62.0	....

[a] Basal medium supplemented with phenylalanine.

[b] Basal medium supplemented with tyrosine.

The crystals decomposed between 230 and 260°C., showing no melting point. The compound was found to react with pyruvic acid in the Strecker reaction[11] with the production of alanine (identified by paper chromatography) and benzaldehyde. The benzaldehyde was identified by precipitation with phenylhydrazine to give the reddish white crystals of the phenylhydrazone of benzaldehyde (m.-p. 154–157°C.).

From this it was indicated that α-phenylglycine might be the substance accumulated. Synthetic *dl*-α-phenylglycine was prepared after the method of Steiger.[12] Paper chromatograms were run on both the unknown and the synthetic *dl*-α-phenylglycine. The spots are identical (table 4). Growth experiments with the unknown and *dl*-α-phenylglycine indicated that the effects on growth are identical (table 5).

Discussion.—The above data show that the phenylalanine and tyrosine requirements of two mutants of *Neurospora crassa* and a strain of *Leuconostoc mesenteroides* may be replaced by α-phenylglycine. One of the mutants, UT 39, accumulates considerable amounts of this substance Whether this should be considered a normal precursor, is open to question. There are at least two possibilities. First, it may be a normal precursor produced in the cell and normally immediately utilized. A second possibility is that it is a product of "shunt metabolism." Thus the normal synthesis of phenylalanine and tyrosine is blocked in UT 39 allowing the accumulation of an unstable precursor which is converted to α-phenylglycine.

The sudden increase in growth of the mutants after a period of days is best explained by postulating that an enzyme system, already present but inefficient in the utilization of α-phenylglycine, is slowly adapted to more efficient usage of available α-phenylglycine. Once it has been adapted, it may perpetuate itself only in the presence of the α-phenylglycine.

The accelerated growth of the mutants after a period of time, in the absence of tyrosine or phenylalanine, resembles certain of the findings of Simmonds[13] with *Escherichia coli* and *Lactobacillus arabinosus*. It is possible that the stimulation that she found with the bacteria may be due to the elaboration of a substance identical with that produced by the mutants used in these studies.

The stimulation of the mutants by intermediates of the Kreb's tricarboxylic acid cycle, the stimulation by Coenzyme I, and the reversal of iodoacetate poisoning by tyrosine and phenylalanine may or may not indicate that these compounds are involved directly in the formation and utilization of α-phenylglycine. A number of hypotheses may be made to explain their stimulatory effect, but it is believed that the data are insufficient at the present time to present a workable hypothesis.

The reason for stimulation of the mutants by α-aminobutyric acid is unknown. A possible explanation is that α-aminobutyric acid functions

directly in the formation of one of the aromatic amino acids or in the conversion of α-phenylglycine to tyrosine or phenylalanine. A more indirect function, however, is equally possible on the basis of the data presented.

It should be noted in conclusion that UT 39 when combined with the pantothenicless mutant 5531 to form the double mutant relieves the requirement for pantothenic acid.[14] The reason for the suppression of the pantothenate requirement by a mutant gene causing the accumulation of α-phenylglycine is not obvious.

The author would like to acknowledge the guidance given by Dr. R. P. Wagner in these studies.

Summary.—Mutants of *Neurospora crassa* requiring tyrosine or phenylalanine for optimal growth were studied. One of these mutants produces an excessive amount of α-phenylglycine which under certain conditions may replace the aromatic amino acid requirements of the mutants and *Leuconostoc mesenteroides*. The significance of this compound, which is here for the first time proved to occur naturally, is discussed.

* Taken, in part, from a thesis presented to the Graduate School of the University of Texas in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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‡ A portion of this work was conducted at the Department of Surgery, Louisiana State University School of Medicine, New Orleans, La.

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THE EFFECT OF OXYGEN CONCENTRATION ON X-RAY-INDUCED CHROMOSOME BREAKAGE IN MAIZE*

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Communicated by M. M. Rhoades, March 24, 1952

The effect of reduced oxygen tension in decreasing the radiosensitivity of biological material to x-rays has been demonstrated in many organisms. Experiments with onion,[1] Tradescantia,[2–4] Drosophila[5] and barley[6] have established that the production of x-ray-induced chromosomal aberrations is influenced by the amount of oxygen present in the cell at the time of irradiation. There are two ways by which the frequency of aberrations can be reduced, (1) by a reduction in the number of effective ionizations and (2) by an increase in the rejoining power of broken ends. The results of earlier investigations have been interpreted as favoring the first hypothesis. It was postulated that x-rays produce chromosome breaks in two ways, by ionization of the molecules of the chromosomes themselves and indirectly by ionization in the vicinity of the chromosome. In the latter case highly active intermediate products such as free radicals or peroxides are liberated which are capable of diffusing to the chromosome and causing breakage. Chromosomes irradiated in the presence of oxygen are thought to be broken by both mechanisms, while in reduced oxygen tension, the indirect effect is minimized.

The experiments described here were conducted to determine if oxygen affects the radiosensitivity of maize chromosomes. Evidence will be presented supporting the hypothesis that the effect of oxygen tension is on the recovery process rather than on the initial breakage mechanism.

Experimental Methods.—Genes *a* (aleurone color) and *sh* (shrunken endosperm) are situated on the long arm of chromosome 3. They are very closely linked giving 0.27% crossing over.[7] Pollen carrying both dominant genes was x-rayed at 1000 r and used to pollinate tester plants homozygous recessive for both genes but carrying all the other factors required for the expression of color in the aleurone, A_2, C and R. The exposures were made with a Collidge self-rectifying tube with a tungsten target, operated at 250 kv. and 15 ma. (3 mm. of Al, inherent filtration).

The pollen to be treated was collected from a number of plants, mixed thoroughly, and divided into two equal parts. One-half the pollen was irradiated in air and the other half in a 100% nitrogen atmosphere. The exposures were made in a lucite chamber. In the case of the nitrogen treatment, the chamber was made airtight and the pollen preflushed with the gas for 3 minutes. A steady flow of nitrogen was passed through the chamber during the $4^1/_2$ minutes of exposure.

Two classes of mutant kernels were observed, whole seed losses where the entire kernel showed only the recessive phenotype, and mosaics which were sectorial for areas of both dominant and recessive tissue. Approximately 3% of the kernels in the former class showed loss of only one of the two genes. All the kernels in the second class were mosaic for both genes.

Since closely linked genes were used in this study it is possible to distinguish the genetic effects due to chromosomal aberrations from those due to gene mutation. Losses of both *A* and *Sh* are due to chromosome breakage. Stadler and Roman[8] have shown that x-rays produce very few, if any, true gene mutations. Thus the chances that both genes *A* and *Sh* mutate simultaneously in the same pollen grain are practically nil. The single gene mutants are due to gene mutations or an internal deletion where one of the breaks occurred in the region between the two genes.

Results and Discussion.—A comparison of irradiation effects produced in nitrogen and in oxygen shows a decrease in the number of mutant kernels by a factor of 3.2 (table 1). This over-all protection value compares well with that obtained with Tradescantia and Drosophila. The data also indicate that there is a significant difference between the relative protection of

TABLE 1

FREQUENCY OF MUTANT KERNELS FOLLOWING IRRADIATION IN AIR AND NITROGEN

	NO. OF SEEDS	WHOLE LOSSES NO.	WHOLE LOSSES %	MOSAICS NO.	MOSAICS %	TOTAL, %
Air	6158	159	2.58	40	0.65	3.23
N_2	4505	43	0.95	3	0.06	1.02
Air/N_2	..	..	2.7	..	9.9	3.2

low oxygen concentration in the two classes of mutant kernels. The whole losses show only 2.7-fold protection while the mosaics show 9.9-fold protection. This difference is significant at the 2% level ($\chi^2 = 5.7$) as determined by the likelihood ratio test and may offer a clue to the mechanism of oxygen protection. If reduced oxygen tension acts only to reduce the number of chromosome breaks, as has been postulated, there should be no difference in the direction found in the ratio of whole losses to mosaics when the exposures are made in oxygen or nitrogen. That a difference of this kind does exist suggests that it is the mechanism of chromosome restitution following breakage which is sensitive to the oxygen tension in the cell at the time of irradiation.

This will be clarified by an analysis of the types of chromosomal aberrations which give rise to the two classes of mutant kernels. Let us consider first breaks limited to the long arm of chromosome 3. The mosaics arise following a break distal to the *A-Sh* region. After chromosomal duplication the two broken ends of the sister strands fuse in endosperm tissue. Due to the chromatid type of breakage-fusion-bridge cycle a mosaic phenotype

results in the endosperm.[9] The whole-loss kernels can arise from two types of aberrations, (1) chromosome breakage proximal to the *A-Sh* region resulting in a terminal deletion which includes the *A* and *Sh* genes and (2) chromosome breakage on both sides of the *A-Sh* region followed by recombination of broken ends in such a way as to result in an interstitial deletion. It is apparent that the mosaic class results from chromosome breakage without reunion (with the exception of a few rings and dicentrics), while a portion of the whole-loss class (i.e., interstitial deletions) results from breakage with reunion of broken ends. If irradiation in nitrogen increases the ability of broken chromosome ends to rejoin, the number of terminal deletions, and hence dicentric chromatids, would be reduced but the interstitial deletions would not be reduced, since they represent a class where chromosomal rejoining had, in fact, occurred. This would result in a more marked protection of low oxygen tension against the mosaic class as compared to the whole losses.

Mosaic kernels may also arise from other aberrations such as centric rings and dicentrics produced by the fusion of the broken ends of two centric chromosomes. These aberrations produce the chromosomal type of breakage-fusion-bridge cycle.[10] In this study it was not possible to distinguish between endosperm mosaics due to the chromatid and chromosome types of cycles. The frequency of occurrence of rings and dicentric chromosomes which contain the *A-Sh* region was determined in another set of experiments involving sporophyte tissue. The marker used in these studies was the minute, cytologically undetectable $a\text{-}x_1$ deficiency found by Stadler and Roman which includes the *A*, *Sh*, and *W* (a chlorophyll gene) loci.[8] The chromatid cycle persists only in the gametophyte and endosperm while the chromosomal type continues in the sporophyte or plant tissue as well.[10] Hence, plants mosaic for anthocyanin and chlorophyll pigmentation must be due to the chromosomal breakage-fusion-bridge cycle. It was found that a dose of 1000 r in air yielded 4 mosaic plants in a population of 5718. Assuming that the frequency of occurrence of ring and dicentric chromosomes is the same in the endosperm and the sporophyte 4 of the 40 endosperm mosaics obtained in the previous set of experiments should be due to these aberrations and 36 to terminal deletions. On the hypothesis that the oxygen effect is exerted on the recovery process, the kernels which are mosaic because of ring and dicentric chromosomes, would not be protected. Hence, the frequency of their occurrence should not be reduced, but perhaps increased, in the nitrogen treatment. Three such mosaics would be expected out of the 4505 kernels in the nitrogen-treated material. This would account for all the mosaics actually found. Thus, there appears to be an even more striking reduction in the occurrence of terminal deletions than the data actually show.

Breakage involving other chromosomes in addition to chromosome 3 does not affect this picture. Let us consider a case of breakage involving the distal region of chromosome 3 and another chromosome. Reunion of the two centric fragments to form a dicentric chromosome is infrequent and has already been considered. Rejoining of the chromosome 3 centric fragment with either acentric fragment results in a phenotypically normal kernel. Only those cases where the broken end of the No. 3 centric fragment remains free result in mosaics. Hence here, also, mosaics result from breakage without reunion. Where the break in chromosome 3 occurred in the region proximal to the *A-Sh* loci only those cases where the acentric No. 3 fragment failed to rejoin with either centric fragment give rise to whole losses. This class of whole losses is reduced in nitrogen treatment as are the mosaics.

Further work is in progress utilizing a chromosome marked throughout its length, which will facilitate distinguishing between interstitial and terminal deletions.

If oxygen is acting on the recovery process it must be an immediate effect, induced at the time of irradiation, and which persists once it is initiated. Giles and Riley[3] have shown the oxygen tension existing in the cell at the time of irradiation to be the important factor. Altering the oxygen tension immediately after irradiation does not alter the frequencies of chromosomal aberrations. Thus it must be postulated that there is a qualitative difference between broken ends of chromosomes, depending upon the amount of oxygen present at the time of breakage. Chromosomes broken in the absence of oxygen are more capable of rejoining than chromosomes broken in the presence of oxygen. McClintock has presented evidence that the behavior of broken ends is "probably related to the method by which the chromosome becomes broken and to physiological conditions surrounding the broken end."[9]

The data presented are not in agreement with the alternative hypothesis that fewer breaks are produced by irradiation in nitrogen. If this were the case the frequency of mosaics to whole losses should be higher in nitrogen than in oxygen. A reduction in the number of breaks would lessen the probability of the occurrence of two breaks in a single chromosome arm, thereby decreasing the frequency of interstitial deletions and hence whole losses. This is the reverse of what has actually been found.

Summary.—Experiments performed on the irradiation of pollen in air and 100% nitrogen atmospheres have demonstrated the effect of low oxygen concentration on the radiosensitivity of maize chromosomes. The differential protection of low oxygen tension against interstitial and terminal deletions suggests that the effect of oxygen is exerted on the recovery process rather than on the initial breakage mechanism.

Acknowledgments.—The author wishes to acknowledge the assistance of Miss Rachel Clark. He is also indebted to Dr. A. W. Kimball for aid in the statistical analysis of the data.

* Work performed under Contract No. W-7405-eng-26 for the Atomic Energy Commission.

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NEW WHITE DWARFS IN THE SOUTHERN HEMISPHERE: SECOND LIST

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Communicated by Harlow Shapley, April 22, 1952

Following up the color survey of proper motion stars in the Southern Hemisphere reported on previously,[1] results have now become available for the zone with declination limits $-55°$ and $-60°$. In the original survey for the detection of proper motions 48 pairs of plates taken with the Harvard 24-inch Bruce telescope were used. These were blinked by the writer and some 6500 stars were found to possess appreciable proper motion. These motions have all been published in part C of the General Catalogue of Proper Motions. Plates were taken for all 48 regions with the same telescope and on a yellow-sensitive emulsion; these have now been compared with the original blue plates and approximate colors for the 6500 proper motion stars determined.

The vast majority of these stars are ordinary red dwarfs, of course, but some 250 stars were singled out as definitely bluer than expected. Narrowing down the limits still further, forty-three stars have been selected from among these 250 as holding out the greatest probability of being genuine white dwarfs. Data for these stars are given in table 1, the several columns of which require no further explanation.

The forty-three stars have been arranged in four groups, as follows:

(*a*) The first group contains the five stars which had previously been identified as white dwarfs from color plates taken at the Cordoba Observa-

TABLE 1

BPM	L	R.A. 1950	DEC.	m	μ	p	I.C.
16115	170-27	0^{h}24^{m}4	−55°42′	14.6	0.585	214°	0^{m}0
16139B	170-14B	27.5	−54 58	15.8	0.340	242	+0.3
18164	182-61	6 15.5	−59 11	13.7	0.301	184	+0.2
26734	210-160	20 18.3	−58 31	16.2	0.511	144	−0.1
27273	212-19	21 15.8	−56 03	15.0	0.450	114	+0.2
16421	171-50	1 08.3	−58 08	14.8	0.122	99	+0.2
16571	172-15	38.7	−55 58	14.6	0.155	100	+0.2
17088	175-34	3 08.5	−56 34	13.4	0.113	76	0.0
18394	184-75	7 01.5	−58 46	14.6	0.187	343	0.0
18615	185-53	40.8	−57 02	15.7	0.074	336	0.0
18972	186-120	8 20.7	−58 32	16.2	0.194	298	+0.1
19652	188-86	9 34.3	−58 43	16.0	0.100	183	−0.2
19738	189-36	50.0	−57 12	14.9	0.188	140	0.0
21641	195-29	12 53.4	−56 09	16.0	0.126	262	+0.3
22038	196-234	13 27.3	−59 56	15.6	0.116	276	+0.2
23550	200-115	15 32.5	−59 35	15.8	0.131	219	+0.1
24047	202-179	16 16.2	−59 09	15.3	0.243	209	−0.3
24723	203-131	17 09.8	−57 34	15.2	0.205	207	0.0
24866	204-118	26.1	−57 53	14.8	0.110	9	+0.2
26536	209-44	20 00.4	−56 12	15.5	0.083	133	+0.1
26744	210-106	20 19.0	−57 21	16.3	0.125	309	+0.2
27891	214-57	22 32.0	−57 31	13.9	0.200	96	0.0
16085	170-31	0 20.2	−55 57	14.8	0.096	206	+0.2
16715	173-50	2 00.0	−58 12	13.6	0.095	193	+0.2
19929	190-21	10 13.2	−55 55	15.2	0.136	298	+0.2
20372	191-28	52.5	−55 46	14.7	0.085	274	+0.2
21018	192-30	11 29.1	−56 24	15.8	0.040	261	+0.1
21094	192-81	11 35.8	−57 32	14.4	0.092	310	+0.1
22046	196-10	13 27.7	−55 14	15.4	0.128	246	+0.2
22855	199-147	14 41.3	−59 17	14.1	0.101	156	+0.2
23593	200-34	15 36.8	−56 12	16.5	0.065	211	+0.2
24290	202-131	16 34.2	−57 34	14.5	0.063	178	+0.2
24342	203-218	16 40.4	−59 43	14.7	0.084	198	+0.2
24537	203-213	55.3	−59 34	14.2	0.087	198	+0.3
24919	204-16	17 30.2	−55 15	12.8	0.110	216	+0.2
26691	210-114	20 14.9	−57 31	12.8	0.076	118	+0.1
26998	211-90	40.1	−58 18	13.9	0.050	126	+0.3
27561	213-30	21 46.2	−56 18	13.9	0.095	146	+0.3
16628	173-64	1 49.0	−58 48	14.0	0.051	141	
16964	175-70	2 52.0	−58 04	15.3	0.052	201	Less
...	180-12	5 22.3	−59 18	15.0	0.102	128	than
26183	208-31	19 27.0	−56 49	16.6	0.031	206	+0.4
26658	210-25	20 10.6	−55 13	16.4	0.199	169	

tory. Four of these were again recognized as white in the present survey but the fifth, L 170-14 B, is the faint companion of a close binary not resolved on the Harvard plates.

(*b*) The second group comprises the seventeen stars for which the probability is very high that they are white dwarfs.

(*c*) The third group contains sixteen further stars; probably at most one-third of these will ultimately prove to be white dwarfs. But while this will remove several stars from the list individually, others from among the 250 are expected to take their place and statistically speaking the stars in this group should be added to the number in group (*b*).

(*d*) The fourth group contains those five stars which were invisible on the yellow plates. Past experience has shown that the chance of their being really white is very high but not much more can be said at the present time.

It is a pleasure to acknowledge the financial support received from the Rumford Fund of the American Academy of Arts and Sciences, which made possible the taking of the yellow plates. The author is further indebted to the Harvard Observatory, and to its director, Dr. Harlow Shapley, for his cooperation in having these plates taken and his continued interest in and support of this program of research.

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*THE SPATIAL CORRELATION OF ELECTRONS IN ATOMS AND MOLECULES II**

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Communicated by R. S. Mulliken, March 31, 1952

1. Introduction.—A discussion of the spatial correlation of electrons of the same spin in atoms and molecules has been given by the author in another paper presented to this Conference.[1] It was there shown that the exclusion principle is a potent factor in keeping electrons of the same spin apart. But to the approximation that the wave function of a many-electron system is represented by a single determinant there is no correlation between electrons of opposite spin. On physical grounds, however, we must expect some correlation between electrons, whether of the same or opposite spin, because of the electrostatic forces between them. This correlation is neglected when electrons of opposite spin are described by a simple product of two similar functions, each involving the coordinates of one electron only. This is done, for example, when the helium atom is de-

scribed approximately by a product of wave functions of the Hartree type, or when the electrons of the hydrogen molecule are described by a product of simple molecule orbital functions. In each case the distribution in space of one electron is assumed to be independent of the position of the other (except for some average of the electrostatic field).

Attempts to improve this simple approximation to the wave function of a pair of electrons of opposite spin have been made by many authors by the method of "configuration interaction." Essentially this method is an attempt to take into account the effect of electrostatic repulsion on spatial correlation, though its physical significance is not always realized nor are its implications discussed. Sometimes the correlation of electrons due to electrostatic repulsion is represented empirically by the introduction of the coordinate r_{12} into the wave function, as Hylleraas did for the helium atom and James and Coolidge[2] for the hydrogen molecule. When this is done, the improvement in the calculation of the energy of the system is most marked, but the method has not proved to be popular because of the awkward form of the wave function and the difficulty of visualizing its properties.

An attempt is made in this paper to introduce the effect of the electrostatic field on spatial correlation in a more direct and logical way. The mutual repulsion of two electrons depends only on their distance apart but since the probability of the occurrence of the electrons in prescribed elements of volume will, in general, be a function of six-coordinate space, it is desirable also to consider the electrostatic repulsion as a function of six dimensional space. The form of this function is well known when both electrons are described in terms of spherical polar coordinates referred to the same center. It is also known when the electrons are described by elliptic coordinates, as is convenient for two center problems. In each case the function is expressed as a series of terms, each of which has the same symmetry in three-dimensional space but has a different symmetry in six-dimensional space. The wave function of the two-electron system has accordingly also been expanded in a series of terms with corresponding symmetry properties. It is then found that the wave equation can be separated into a number of component parts, each of which has the same symmetry in six-dimensional space. This leads to a simplification of the general problem and to a closer understanding of the way the electrons affect each other in their spatial distribution.

2. *Helium and Similar Atomic Systems in the Ground State.*—For a system of two electrons, each of which moves in the same central field $V(r)$, the wave equation in atomic units is

$$(H_1 + H_2 + 1/r_{12})\psi(1, 2) = E\psi(1, 2), \qquad (2.1)$$

where H as usual includes the kinetic energy operator and $V(r)$. The ex-

pansion of $1/r_{12}$ in spherical polar coordinates is

$$1/r_{12} = \sum_{\tau} U_{\tau} P_{\tau}(\cos \theta_{12}), \tag{2.2}$$

where

$$U_{\tau} = r_1^{\tau}/r_2^{\tau+1}, \tag{2.3}$$

when r_1 is smaller than r_2, and P_{τ} is the usual Legendre function. The latter function can also be expressed in the usual way as a sum of terms involving the Legendre functions of cos θ_1, cos θ_2 and cos $\nu(\phi_1 - \phi_2)$. For the ground state of helium and similar systems in ^1S states this is not necessary, as the distribution in three-dimensional space is spherically symmetrical and the wave function can only involve the radial coordinates and θ_{12}. It is accordingly convenient to express the wave function in a form similar to (2.3), viz.

$$\psi(1, 2) = \sum_{\tau} \psi_{\tau}(1, 2) P_{\tau}(\cos \theta_{12}). \tag{2.4}$$

When a function of this type is substituted in the wave equation (2.1) and use is made of the fact that a product of two Legendre functions can be expressed as a linear sum of Legendre functions, the wave equation separates out into terms each of which contains a Legendre function of the same order, say P_{τ}. Multiplying the equation by P_{τ} and integrating over the variable θ_{12}, all terms vanish except one set. In this way a set of coupled equations for the functions $\psi_{\tau}(1, 2)$ are obtained as follows

$$\left\{ -\frac{1}{2}\left[\frac{1}{r_1^2}\frac{\partial}{\partial r_1}\left(r_1^2\frac{\partial}{\partial r_1}\right) + \frac{1}{r_2^2}\frac{\partial}{\partial r_2}\left(r_2^2\frac{\partial}{\partial r_2}\right)\right] + \frac{\tau(\tau+1)}{2}\left[\frac{1}{r_1^2} + \frac{1}{r_2^2}\right] + V(r_1) + V(r_2)\right\} \psi_{\tau} + \sum_{s,t} c_{s,t}^{1/2(s+t-\tau)} U_s \psi_t = E\psi_{\tau} \tag{2.5}$$

where $c_{s,t}^{(\tau)}$ are numerical coefficients, whose value is zero if τ is negative or nonintegral. In particular the equation for ψ_0, the first term of the series (2.4) proves to be

$$\left\{ -\frac{1}{2}\left[\frac{1}{r_1^2}\frac{\partial}{\partial r_1}\left(r_1^2\frac{\partial}{\partial r_1}\right) + \frac{1}{r_2^2}\frac{\partial}{\partial r_2}\left(r_2^2\frac{\partial}{\partial r_2}\right)\right] + V(r_1) + V(r_2)\right\} \psi_0 + \sum_{\tau=0}^{\infty}\left\{U_{\tau}/(2\tau+1)\right\} \psi_{\tau} = E\psi_0 \tag{2.6}$$

The Hartree solution for helium, or any method of representation of the $(1s)^2$ type, involves two approximations to (2.6). Firstly all terms in the expansion of $1/r_{12}$ other than U_0 are neglected and secondly ψ_0 (1,2) is as-

sumed to be expressible as a product of a function of the coordinates of electron 1 and the same function of the coordinates of electron 2.

The calculations of Taylor and Parr[3] concerning the energy of the ground state of the helium atom, reported to this conference, are seen to fall within the above scheme. They express the wave function as a sum of terms, the first of which can be denoted symbolically by $(1s, 1s')$, the second by a sum over $2p$ functions, the third by a sum over $3d$ functions and the fourth by a sum over $4f$ functions. These correspond to the first few terms of (2.4) except that the radical part of each term is expressed as a product of hydrogen-like functions with appropriate values of Z, the effective nuclear charge. In spite of this limitation the convergence of the calculated energy toward the observed one, as the wave function is represented by one, two, three and four terms, is remarkably good. The best value of the calculated total energy differs by only about one-eighth of one per cent from the observed value. Even a two-term expansion suffices to give an accuracy of about one-sixth of one per cent.

From the wave function obtained by Taylor and Parr some information can be derived about the extent of spatial correlation in helium. This may be of two types. Firstly one electron may tend to be "out" when the other is "in," and secondly one electron may tend to be on the opposite side of the nucleus from the other. It appears from the first term of their wave function (of the $1s, 1s'$ type) that when r_1 is small, the most probable value of r_2 is about 0.34 Å, whereas when r_1 is large, the most probable value of r_2 is reduced to 0.23 Å. This indicates the extent of *in-out* correlation for a function of this type.

Some idea of the extent of angular correlation may be obtained from the first two terms of the wave function. When there is no angular correlation at all, the average value of the angle subtended by the electrons at the nucleus is 90°. The deviation from this value in helium does not seem to be large. It has a maximum when r_1 and r_2 are both about 0.6 Å and is then about 6°. When r_1 and r_2 have their most probable values (about 0.3 Å) the deviation is about 4°. The average deviation for all values of r_1 and r_2 is slightly less than this.[4]

3. The Hydrogen Molecule.—For the hydrogen molecule and similar homonuclear systems it is convenient to use elliptic coordinates. If ξ_1, and ξ_2 are the coordinates for the confocal ellipsoids, η_1 and η_2 those for the hyperboloids and ϕ_{12} the angle between two azimuthal planes, then $1/r_{12}$ can be expressed in terms of them by the Neumann expansion. This may be put in the form

$$c/2r_{12} = \sum_{\tau=0}^{\infty} \sum_{\nu=0}^{\tau} D_{\tau,\nu}(\xi_1, \xi_2)\, P_\tau^\nu(\eta_1) P_\tau^\nu(\eta_2) \cos \nu\phi_{12} \qquad (3.1)$$

where c is the distance between the two centers and $D_{\tau,\nu}$ is a function in-

volving only ξ_1 and ξ_2. Each term is symmetrical for exchanges of the electrons and the symmetry properties of the various terms in the multidimensional space are determined by the factors in η_1, η_2 and ϕ_{12}. These may be symmetrical or antisymmetrical for changes of sign of η_1 or of η_2, and there is periodic change of sign in ϕ_{12}.

The wave function for the ground state of the hydrogen molecule, being a $^1\Sigma_g$ state, must be symmetrical in the space coordinates of the electrons and so it is convenient to express it as a series of the same type as (3.1), viz.

$$\psi(1, 2) = \sum_{\nu=0}^{\infty} (\psi_{s,\nu} + \psi_{a,\nu}) \cos \nu\phi_{12} \qquad (3.2)$$

where $\psi_{s,\nu}$ and $\psi_{a,\nu}$ are each functions of the coordinates of electrons 1 and 2, and have the same symmetry properties as the terms in (3.1), being either even in both η_1 and η_2 or odd in both η_1 and η_2.

When $\psi(1, 2)$ is substituted in the wave equation, the terms arising from $(1/r_{12})\psi(1, 2)$ can be arranged according to their symmetry types in the multidimensional space. All the terms which contain $\cos \nu\phi_{12}$ can be equated separately to zero. Further the resulting set can again be subdivided into those of symmetry $\psi_{s,\nu}$ and those of symmetry $\psi_{a,\nu}$. Since the expression must be identically zero for all points in the $(\xi_1, \xi_2, \eta_1, \eta_2, \phi_{12})$ space and cannot change when η_1 is changed to $-\eta_1$, the parts with $\psi_{s,\nu}$ and $\psi_{a,\nu}$ symmetry must separately vanish. Thus a set of coupled equations can be obtained for the set of functions $\psi_{s,\nu}$ and $\psi_{a,\nu}$, ν ranging through all integral values. These have been given elsewhere by Lennard-Jones and Pople.[5]

Although direct solutions of these equations may not be possible at present, they form a useful background against which to examine the limitations of approximate functions. Thus the simple molecular orbital representation of the wave function in the form $f_{s,0}(1)f_{s,0}(2)$ involves the neglect of all parts of $\psi(1, 2)$ which depend on ϕ_{12} and also of the part which has symmetry $\psi_{a,0}$. It also expresses the wave function as a product and so automatically cuts out spatial correlation. A better approximation to the wave function would be given by

$$\psi(1, 2) = f_{s,0}(1)f_{s,0}(2) - f_{a,0}(1)f_{a,0}(2), \qquad (3.3)$$

where $f_{a,0}$ is a one-electron function independent of ϕ and antisymmetric in the plane $\eta = 0$. The negative sign in (3.3) is given because this proves to be necessary if $f_{a,0}$ is to be real everywhere. The condition that the energy should be a minimum leads to differential equations which $f_{s,0}$ and $f_{a,0}$ must satisfy.[6]

Even the simple function (3.3) involves spatial correlation to some extent. For if the position of electron 1 is fixed, the distribution of electrons depends on the square of a linear expression in $f_{s,0}(2)$ and $f_{a,0}(2)$. This

is not symmetric in the plane of symmetry and so there is a greater probability of finding the electron nearer one nucleus than the other; in fact, the electrons tend to be on opposite sides of the plane of symmetry. By writing

$$\chi_A = (f_s + f_a)/\sqrt{2},\ \chi_B = (f_s - f_a)/\sqrt{2} \tag{3.4}$$

the approximate form (3.3) is seen to be equivalent to

$$\psi(1, 2) = \chi_A(1)\chi_B(2) + \chi_B(1)\chi_A(2). \tag{3.5}$$

The Heitler and London wave function and all similar functions involving directed atomic orbitals are seen to be particular cases of (3.5).

A better approximation to $\psi(1, 2)$ can be obtained by including in (3.3) terms in $\cos \nu\phi_{12}$. This leads to a generalization of (3.5) in the form

$$\psi(1, 2) = \sum_{\nu=0}^{\infty} \{\chi_{A,\nu}(1)\chi_{B,\nu}(2) + \chi_{B,\nu}(1)\chi_{A,\nu}(2)\} \cos \nu\phi_{12} \tag{3.6}$$

An expression of this kind implies not only correlation with respect to the plane of symmetry but correlation also in the azimuthal angle. It is still lacking in one respect. It does not allow for correlation of the "in-out" type, discussed above for helium. This would be included in the correct solution for $\psi(1, 2)$ even in the first term of the expansion (3.2).

Some indication of the effect of correlation on the energy of the hydrogen molecule for a wave function of the type (3.6) can be found from an analysis of some results recently published by Hirschfelder and Linnett (1950).[7] They use an approximate wave function built up from atomic orbitals of the hydrogen 1*s* type, ϕ_A and ϕ_B, with adjustable screening constant. The first term, independent of the angle ϕ_{12}, is then

$$\phi_A(1)\phi_B(2) + \phi_A(2)\phi_B(1) + \gamma\{\phi_A(1)\phi_A(2) + \phi_B(1)\phi_B(2)\} \tag{3.7}$$

where γ is an adjustable parameter. The second term of (3.6) is approximated to by taking $\chi_{A,1}$ and $\chi_{B,1}$ to be multiples of

$$\phi_A r_A \sin \theta_A;\quad \phi_B r_B \sin \theta_B$$

which simulate atomic *p* functions. This part contains a numerical factor α which is adjusted to give the best value of the energy.

By giving α and γ various values, approximations can be obtained to the types of wave function dealt with above. Thus we have the following special cases:

(i) *Function 1* ($\alpha = 0$, $\gamma = 1$): This corresponds to a product of simple molecular orbitals.

(ii) *Function 2* ($\alpha = 0$, γ adjustable): This is usually described as the electron pair function with ionic terms. It is also of the form (3.3) pro-

vided $f_{s,0}$ and $f_{a,0}$ are expressed in terms of atomic orbitals. The optimum value of γ is 0.267.

(iii) *Function 3* (α and γ adjustable): This takes in azimuthal terms in addition to those of Function 2. The optimum values of α and γ are -0.054 and 0.273, respectively.

Using the tables given by Hirschfelder and Linnett,[7] the various contributions to the total energy for these three functions can be calculated. These have been calculated for an effective nuclear charge 1.19 and a value of 1.47 for the internuclear distance, these being close to those given by them. The results are shown in the table, the contributions to the energy of the molecule being expressed in kilocalories per mole relative to the separated atoms.

CONTRIBUTIONS TO THE ENERGY OF THE HYDROGEN MOLECULE RELATIVE TO SEPARATED ATOMS (IN KCAL./MOLE)

	KINETIC ENERGY	ELECTRON-ELECTRON POTENTIAL ENERGY	NUCLEAR-NUCLEAR POTENTIAL ENERGY	ELECTRON-NUCLEAR POTENTIAL ENERGY	TOTAL
Function 1	71	405	426	−981	−79
Function 2	87	379	426	−985	−92
Function 3	87	372	426	−981	−96

The table shows up the improvement in the calculated binding energy as functions of different symmetries are introduced into the total wave function. The second column shows how the average electrostatic energy of the electrons is progressively decreased, thus emphasizing the importance of spatial correlation. The first and last columns would be equal in magnitude if the virial theorem were satisfied. The divergence is doubtless due to the fact that the values of the effective nuclear charge and the interatomic distance used were not exactly the optimum ones.

* This paper was prepared in connection with a conference on "Quantum-Mechanical Methods in Valence Theory" held at Shelter Island, September 7–10, 1951, under the auspices of the NATIONAL ACADEMY OF SCIENCES (see "Summary of the Conference" in a subsequent issue of these PROCEEDINGS).

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[4] A fuller discussion of the helium problem with particular reference to spatial correlation is to be published elsewhere by the author and J. A. Pople (*Phil. Mag.*, May, 1952).

[5] Lennard-Jones, J. E., and Pople, J. A., *Proc. Roy. Soc.*, **A, 210**, 190 (1951).

[6] The equations have been given elsewhere (Lennard-Jones and Pople, *Proc. Roy. Soc.*, **A, 210**, 190 (1951)), where also a discussion of approximate methods of dealing with the hydrogen molecule is to be found.

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GROWTH OF ANALYTIC FUNCTIONS ALONG A LINE

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Communicated by S. Bochner, March 29, 1952

Let $f(z)$ be regular and of exponential type less than π in the half plane $x \geqq 0$, and let $\{\lambda_n\}$ be an increasing positive sequence of unit density such that $\lambda_{n+1} - \lambda_n \geqq \delta > 0$. It is known that under suitable restrictions and with a suitable measure of growth, the growth of the sequence $\{f(\lambda_n)\}$ determines the growth of $f(x)$ for real $x > 0$. I have established the following results.

Let $\epsilon(x)$ be a concave increasing differentiable function such that $\epsilon(x) = o(x)$, $\epsilon'(x) \leqq \epsilon(x)/x < 1$ and $\log x = o(\epsilon(x))$. Let $\{\lambda_n\}$ satisfy the further condition that $|\lambda_n - n| \leqq \epsilon(n)$.

THEOREM 1. *If*

$$\limsup_{n \to \infty} \frac{\log|f(\lambda_n)|}{\epsilon(\lambda_n)} < \infty$$

then

$$\limsup_{x \to \infty} \frac{\log|f(x)|}{\epsilon(x)} < \infty.$$

THEOREM 2. *If*

$$\limsup_{n \to \infty} \frac{\log|f(\pm\lambda_n)|}{\epsilon(\lambda_n)} = -B < 0,$$

where B is sufficiently large, then

$$\limsup_{x \to \infty} \frac{\log|f(x)|}{\epsilon(x)} < 0.$$

THEOREM 3. *If $f(z)$ is entire and of zero exponential type, $\{f(\pm\lambda_n)\}$ is bounded, and $\epsilon(x)$ satisfies the additional condition*

$$(*) \qquad \int_0^\infty x^{-2}\epsilon(x)\,dx < \infty$$

then $f(z)$ is a constant.

THEOREM 4. *If $f(z)$ is entire and of zero exponential type and $f(\pm\lambda_n) = O\{\exp(\lambda_n^\rho)\}$, $0 < \rho < 1$, then $f(z)$ is of order not exceeding ρ.*

Theorem 1 interpolates between, but does not include, the theorem of V. Bernstein[1] in which $n - \lambda_n = o(n)$ and the conclusion is

$$\limsup_{x \to \infty} x^{-1} \log|f(x)| = \limsup_{n \to \infty} \lambda_n^{-1} \log|f(\lambda_n)|;$$

and the theorem of Duffin and Schaeffer[2] in which $\epsilon(x)$ is a constant, i.e., $|n - \lambda_n| \leqq L$, and the conclusion is that $f(x)$ is bounded if $\{f(\lambda_n)\}$ is bounded. Agmon[3] has treated similar rates of growth of the function but with a more restrictive condition on $\{\lambda_n\}$. As far as concerns the magnitude of $\epsilon(x)$, Theorem 3 is an improvement on Levinson's generalization[4] of Pólya's theorem in which $\lambda_n = n$. Theorem 3 covers, for example, $\epsilon(x) = x/\{\log x \ (\log \log x)^2\}$ for large x, which is not covered by Levinson's theorem. Levinson has shown that the convergence of (*) is essential. Levinson's work is, however, more general in allowing complex λ_n and a weaker spacing condition than $|\lambda_n - \lambda_m| \geqq \delta$, $n \neq m$.

Theorems 1 and 2 have applications to the study of the behavior of power series on the circle of convergence, to Fourier series and to Tauberian theorems.

The theorems are proved by a rather straightforward, but not entirely obvious, process[5] of comparing the given function with an interpolation series based on the points λ_n. This requires sufficiently precise estimates of an entire function with zeros at $\pm\lambda_n$, and similar functions, and it is these estimates which represent the essential novelty of the investigation reported here. The calculations are rather lengthy and will be published elsewhere. The essential estimates are as follows. Let

$$\phi(z) = \prod_{n=1}^{\infty} (1 - z^2/\lambda_n^2);$$

then $|\phi(x + iy)| \leqq Ae^{A\epsilon(x)}$ for $|y| \leqq 1$, and $|\phi'(\lambda_n)| \geqq Ae^{-A\epsilon(\lambda_n)}$, where A denotes various positive constants. If $\epsilon(x)$ satisfies the additional condition (*), then there is an entire function $\psi(z)$ of arbitrarily small positive exponential type such that $\psi(0) = 1$ and $\log|\psi(x)| \leqq -A\epsilon(|x|)$.

* Fellow of the John Simon Guggenheim Memorial Foundation.

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[5] Cf. Boas, R. P., and Schaeffer, A. C., "A Theorem of Cartwright," *Duke Math. J.*, **9**, 879–883 (1942); Korevaar, J., "Approximation and Interpolation Applied to Entire Functions," Amsterdam thesis, 1949; Levin, B., "O funktsiyah konechnoi stepeni, ogranichennyh na posledovatel'nosti tochek," *Doklady Akad. Nauk SSSR (N. S.)*, **65**, 265–268 (1949).

GENERALIZED RIEMANN SPACES, II*

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Communicated March 17, 1952

By definition a generalized Riemann space of coordinates x^i ($i = 1, \ldots, n$) is one with which is associated an asymmetric tensor g_{ij}, whose determinant is not identically zero, and coefficients Γ^h_{ij} of an affine connection asymmetric in the subscripts. It is understood that the coefficients Γ^h_{ij} and $\Gamma'^{\gamma}_{\alpha\beta}$ in coordinate systems x^i and x'^{α} are related by the equations

$$\frac{\partial^2 x^h}{\partial x'^{\alpha}\partial x'^{\beta}} + \Gamma^h_{ij}\frac{\partial x^i}{\partial x'^{\alpha}}\frac{\partial x^j}{\partial x'^{\beta}} = \Gamma'^{\gamma}_{\alpha\beta}\frac{\partial x^h}{\partial x'^{\gamma}} \tag{1}$$

In this equation, and throughout the paper the summation convention indicated by a repeated index, such as i, j and γ in (1) is used.

Interchanging α and β in (1), we write the resulting equation in the form

$$\frac{\partial^2 x^h}{\partial x'^{\beta}\partial x'^{\alpha}} + \Gamma^h_{ji}\frac{\partial x^i}{\partial x'^{\alpha}}\frac{\partial x^j}{\partial x'^{\beta}} = \Gamma'^{\gamma}_{\beta\alpha}\frac{\partial x^h}{\partial x'^{\gamma}}. \tag{2}$$

One-half the sum and one-half the difference of equations (1) and (2) are the respective equations

$$\frac{\partial^2 x^h}{\partial x'^{\alpha}\partial x'^{\beta}} + \Gamma^h_{\underset{\sim}{ij}}\frac{\partial x^i}{\partial x'^{\alpha}}\frac{\partial x^j}{\partial x'^{\beta}} = \Gamma'^{\gamma}_{\alpha\beta}\frac{\partial x^h}{\partial x'^{\gamma}}, \tag{3}$$

and

$$\Gamma^h_{\underset{\sim}{ij}}\frac{\partial x^i}{\partial x'^{\alpha}}\frac{\partial x^j}{\partial x'^{\beta}} = \Gamma'^{\gamma}_{\underset{\sim}{\alpha\beta}}\frac{\partial x^h}{\partial x'^{\gamma}}, \tag{4}$$

where Γ^h_{ij} and $\Gamma'^{\gamma}_{\alpha\beta}$ are the symmetric parts of Γ^h_{ij} and $\Gamma'^{\gamma}_{\alpha\beta}$, and $\Gamma^h_{\underset{\sim}{ij}}$ and $\Gamma'^{\gamma}_{\underset{\sim}{\alpha\beta}}$ the skew-symmetric parts. This notation is used throughout this paper. Thus g_{ij} and $g_{\underset{\sim}{ij}}$ are the symmetric and skew-symmetric parts of g_{ij}. From (4) it follows that $\Gamma^h_{\underset{\sim}{ij}}$ are components of a tensor.

The quantities

$$g_{ij,\,k} - g_{hj}\Gamma^h_{ik} - g_{ih}\Gamma^h_{jk}, \tag{5}$$

where $g_{ij,\,k}$ denotes the derivative of g_{ij} with respect to x^k (a notation used throughout this paper) are the components of a tensor, being the covariant derivative of g_{ij} in terms of the Γ's.[1] Denoting by a_{ijk} the components of a tensor to be specified later, we consider the equation

$$g_{ij,\,k} = g_{hj}\Gamma^h_{ik} + g_{ih}\Gamma^h_{jk} + a_{ijk}. \tag{6}$$

From this equation we have

$$g_{ij,\,k} = g_{hj}\Gamma^h_{ik} + g_{ih}\Gamma^h_{jk} + a_{ijk} \tag{7}$$

where a_{ijk} is the part of a_{ijk} symmetric in the first two indices, that is, $a_{ijk} = \frac{1}{2}(a_{ijk} + a_{jik})$. We choose a_{ijk} so that

$$a_{ijk} + g_{hj}\Gamma^h_{ij} + g_{ih}\Gamma^h_{jk} = 0, \tag{8}$$

in which case equation (7) becomes

$$g_{ij,\,k} = g_{hj}\Gamma^h_{ik} + g_{ih}\Gamma^h_{jk} \tag{9}$$

from which it follows that[2]

$$\Gamma^h_{ik} = \left\{ {h \atop ik} \right\}, \tag{10}$$

where $\left\{ {h \atop ik} \right\}$ is the Christoffel symbol of the second kind in terms of the g_{ij}'s; and similarly for all the parts of the Γ's symmetric in the subscripts.

In consequence of (8) equation (6) reduces to

$$g_{ij,\,k} = g_{hj}\Gamma^h_{ik} + g_{ih}\Gamma^h_{jk} + g_{hj}\Gamma^h_{ik} + g_{ih}\Gamma^h_{jk} + a_{ijk}$$

where a_{ijk} is the part of a_{ijk} skew-symmetric in the first two indices. When equation (9) is subtracted from this equation, one obtains

$$g_{ij/k} = g_{hj}\Gamma^h_{ik} + g_{ih}\Gamma^h_{jk} + a_{ijk} \tag{11}$$

where $g_{ij/k}$ is the covariant derivative of g_{ij} in terms of the Christoffel symbols, that is[3]

$$g_{ij/k} = g_{ij,\,k} + g_{hj}\left\{ {h \atop ik} \right\} + g_{ih}\left\{ {h \atop jk} \right\} \tag{12}$$

From equations (11) we have

$$\tfrac{1}{2}(g_{ij/k} + g_{jk/i} + g_{ki/j}) = g_{hj}\Gamma^h_{ik} + g_{hj}\Gamma^h_{kj} + g_{hk}\Gamma^h_{ji} + \tfrac{1}{2}(a_{ijk} + a_{jki} + a_{kij}).$$

We replace this equation by the equations

$$g_{hj}\Gamma^h_{ik} + \tfrac{1}{2}a_{kij} + c_{ikj} = g_{hj}\Gamma^h_{kj} + \tfrac{1}{2}a_{jki} + c_{kji} = g_{hk}\Gamma^h_{ji} + \tfrac{1}{2}a_{ijk} + c_{jik} = \tfrac{1}{6}(g_{ij/k} + g_{jk/i} + g_{ki/j}) \tag{13}$$

where the c's are components of a tensor skew-symmetric in the first two indices such

$$c_{ikj} + c_{kji} + c_{jik} = 0. \tag{14}$$

Substituting from (13) in equation (11) we have in consequence of (14)

$$c_{ijk} = {}^1/_3(2g_{\underset{\smile}{ij}/k} + g_{\underset{\smile}{kj}/i} + g_{\underset{\smile}{ik}/j}) - a_{\underset{\smile}{ijk}} + {}^1/_2(a_{\underset{\smile}{jki}} + a_{\underset{\smile}{kij}}). \quad (15)$$

These expressions satisfy (14). From (13) and (15) we obtain

$$g_{\underset{\smile}{hk}}\Gamma^h_{\underset{\smile}{ij}} = {}^1/_2(g_{\underset{\smile}{ij}/k} + g_{\underset{\smile}{kj}/i} + g_{\underset{\smile}{ik}/j}) - {}^1/_2(a_{\underset{\smile}{ijk}} + a_{\underset{\smile}{kji}} + a_{\underset{\smile}{ikj}}). \quad (16)$$

If a_{ijk} is chosen so that $a_{\underset{\smile}{ijk}} = 0$, we have

$$g_{\underset{\smile}{hk}}\Gamma^h_{\underset{\smile}{ij}} = {}^1/_2(g_{\underset{\smile}{ij}/k} + g_{\underset{\smile}{kj}/i} + g_{\underset{\smile}{ik}/j}). \quad (17)$$

If on the other hand we choose

$$a_{ijk} = g_{\underset{\smile}{hj}}\Gamma^h_{\underset{\smile}{ki}} + (g_{ih} + g_{\underline{ih}})\Gamma^h_{\underset{\smile}{kj}}, \quad (18)$$

the condition (8) is satisfied, and

$$a_{\underset{\smile}{ijk}} = g_{jh}\Gamma^h_{\underset{\smile}{ik}} - g_{ih}\Gamma^h_{\underset{\smile}{jk}}$$

In this case (16) becomes

$$g_{\underline{hk}}\Gamma^h_{\underset{\smile}{ij}} = {}^1/_2(g_{\underset{\smile}{ij}/k} + g_{\underset{\smile}{kj}/i} + g_{\underset{\smile}{ik}/j}). \quad (19)$$

If again we choose

$$a_{ijk} = 2g_{\underline{ih}}\Gamma^h_{\underset{\smile}{kj}},$$

the condition (8) is satisfied, and in this case we have from (16)

$$g_{hk}\Gamma^h_{\underset{\smile}{ij}} = {}^1/_2(g_{\underset{\smile}{ij}/k} + g_{\underset{\smile}{kj}/i} + g_{\underset{\smile}{ik}/j}). \quad (20)$$

When the condition of integrability

$$\frac{\partial}{\partial x'^\gamma}\left(\frac{\partial^2 x^h}{\partial x'^\alpha \partial x'^\beta}\right) = \frac{\partial}{\partial x'^\beta}\left(\frac{\partial^2 x^h}{\partial x'^\alpha \partial x'^\gamma}\right)$$

of equations (1) is reduced by means of equations of the form (1), one finds that the quantities Γ^h_{ijk} defined by

$$\Gamma^h_{ijk} = \Gamma^h_{ij,\,k} - \Gamma^h_{ik,\,j} + \Gamma^l_{ij}\Gamma^h_{lk} - \Gamma^l_{ik}\Gamma^h_{lj} \quad (21)$$

are components of a tensor. In like manner from the condition of integrability

$$\frac{\partial}{\partial x'^\gamma}\left(\frac{\partial^2 x^h}{\partial x'^\alpha \partial x'^\beta}\right) = \frac{\partial}{\partial x'^\alpha}\left(\frac{\partial^2 x^h}{\partial x'^\gamma \partial x'^\beta}\right)$$

of equation (1) one finds that the quantities $\bar{\Gamma}^h_{ijk}$ defined by

$$\bar{\Gamma}^h_{ijk} = \Gamma^h_{ij,\,k} - \Gamma^h_{kj,\,i} + \Gamma^l_{ij}\Gamma^h_{kl} - \Gamma^l_{kj}\Gamma^h_{il} \quad (22)$$

are components of a tensor.

Interchanging α and β in the above conditions of integrability and applying them to equation (2), one obtains the respective tensors

$$*\Gamma^h_{ijk} = \Gamma^h_{ji,\,k} - \Gamma^h_{jk,\,i} + \Gamma^l_{ji}\Gamma^h_{lk} - \Gamma^l_{jk}\Gamma^h_{li}, \tag{23}$$

$$*\bar{\Gamma}^h_{ijk} = \Gamma^h_{ji,\,k} - \Gamma^h_{ki,\,j} + \Gamma^l_{ji}\Gamma^h_{kl} - \Gamma^l_{ki}\Gamma^h_{jl}. \tag{24}$$

In consequence of (10) equation (21) may be given the form

$$\Gamma^h_{ijk} = -R^h_{ijk} + \Gamma^h_{ij/k} - \Gamma^h_{ik/j} + \Gamma^l_{ij}\Gamma^h_{lk} - \Gamma^l_{ik}\Gamma^h_{lj}, \tag{21'}$$

where $\Gamma^h_{ij/k}$ is the covariant derivative of Γ^h_{ij} in terms of the Christoffel symbols, and R^h_{ijk} is the Riemann tensor defined by[4]

$$R^h_{ijk} = \left\{ {h \atop ik} \right\}_{,\,j} - \left\{ {h \atop ij} \right\}_{,\,k} + \left\{ {l \atop ik} \right\}\left\{ {h \atop lj} \right\} - \left\{ {l \atop ij} \right\}\left\{ {h \atop lk} \right\}. \tag{25}$$

In like manner equations (22), (23), and (24) may be given the forms

$$\bar{\Gamma}^h_{ijk} = -R^h_{jik} + \Gamma^h_{ij/k} - \Gamma^h_{kj/i} + \Gamma^l_{ij}\Gamma^h_{kl} - \Gamma^l_{kj}\Gamma^h_{il}, \tag{22'}$$

$$*\Gamma^h_{ijk} = -R^h_{jik} + \Gamma^h_{ji/k} - \Gamma^h_{jk/i} + \Gamma^l_{ji}\Gamma^h_{lk} - \Gamma^l_{jk}\Gamma^h_{li}, \tag{23'}$$

$$*\bar{\Gamma}^h_{ijk} = -R^h_{ijk} + \Gamma^h_{ji/k} - \Gamma^h_{ki/j} + \Gamma^l_{ji}\Gamma^h_{kl} - \Gamma^l_{ki}\Gamma^h_{jl}. \tag{24'}$$

One-half the sum of the tensors (21) and (22) is the tensor

$$T^h_{ijk} = \Gamma^h_{ij,\,k} - {}^1/_2(\Gamma^h_{ik,\,j} + \Gamma^h_{kj,\,i}) + \Gamma^l_{ij}\Gamma^h_{lk} - {}^1/_2(\Gamma^l_{ik}\Gamma^h_{lj} + \Gamma^l_{kj}\Gamma^h_{il}).$$

Contracting for h and k, we have

$$T_{ij} = \Gamma^h_{ij,\,h} - {}^1/_2(\Gamma^h_{ih,\,j} + \Gamma^h_{hj,\,i}) + \Gamma^l_{ij}\Gamma^h_{lh} - \Gamma^l_{ih}\Gamma^h_{lj} - {}^1/_2(\Gamma^h_{ih,\,j} - \Gamma^h_{jh,\,i}).$$

The expression in the second parentheses being a tensor, the other part of the right-hand member is a tensor; it is the Einstein tensor R_{ij}.[5]

From (21') and (24') one has the tensor

$${}^1/_2(\Gamma^h_{ijk} - *\bar{\Gamma}^h_{ijk}) = \Gamma^h_{ij/k} - \Gamma^h_{ik/j},$$

and from (22') and (23') the tensor

$${}^1/_2(\bar{\Gamma}^h_{ijk} - *\Gamma^h_{ijk}) = \Gamma^h_{ij/k} - \Gamma^h_{kj/i}.$$

* See these PROCEEDINGS, **37**, 311–315 (1951).

[1] Eisenhart, L. P., "Non-Riemannian Geometry," *Am. Math. Soc. Colloquium Publications*, **5**, p. 5.

[2] *R. G.*, p. 17. This refers to the author's Riemannian Geometry, Princeton University Press.

[3] *R. G.*, p. 27.

[4] *R. G.*, p. 19.

[5] Einstein, A., *The Meaning of Relativity*, 3rd ed. (revised), Princeton University Press, p. 114.

ON THE STATISTICAL THEORY OF TURBULENT DIFFUSION*

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Communicated by J. von Neumann, April 30, 1952

1. Introduction.—The fundamental equations of turbulent diffusion have been first established by G. I. Taylor. In his paper on diffusion by continuous movements,[1] he considers the one-dimensional case of diffusion and finds the variance of the distance through which the particles have dispersed from a point-source. This variance is expressed as a function of the dispersion time and of two statistical characteristics of the turbulent field: the variance of the turbulent velocity and the Lagrangian correlation coefficient. Taylor's equations can be used as an approximation in the case of the three-dimensional turbulent diffusion from a continuous source in a fluid flow, when the turbulent velocities are small compared to the mean flow velocity.[2] However, in many applications to fluid dynamics, meteorology, oceanography, chemistry, etc., turbulent velocities are of the same order of magnitude as the mean velocity or even of a larger order. The main purpose of this communication is to consider the turbulent diffusion in a field of homogeneous and isotropic turbulence when the turbulent velocities cannot be assumed small as compared to the mean velocity.

2. Fundamental Equations of Turbulent Diffusion.—Let us consider within an incompressible turbulent fluid at rest, of homogeneous and isotropic turbulence, a fluid particle which at an initial time 0 is placed at the origin of the coordinate axes. Let $u(t)$, $v(t)$ and $w(t)$ be the x-, y- and z-components of the instantaneous velocity of the particle. After a time t, the coordinates of the particle are

$$x = \int_0^t u(\alpha)d\alpha;\ y = \int_0^t v(\alpha)d\alpha;\ z = \int_0^t w(\alpha)d\alpha.$$

The coordinates x, y and z are random variables[3] and we shall investigate their variances as functions of the dispersion time t. The mean values of these coordinates $\bar{x} = \bar{y} = \bar{z} = 0$ and their variances are functions of the dispersion time, and since the turbulence is homogeneous and isotropic, $\overline{x^2} = \overline{y^2} = \overline{z^2} = \psi(t)$.

We shall determine $\psi(t)$ as a function of the turbulence characteristics. We find

$$\overline{y^2} = \overline{\left[\int_0^t v(\alpha)d\alpha\right]^2} = \int_0^t \int_0^t \overline{v(\alpha_1)v(\alpha_2)}d\alpha_1\, d\alpha_2$$

and introducing the Lagrangian correlation coefficient[4] $R_h(h) = \overline{v(t)v(t+h)}/\overline{v^2}$,

$$\overline{y^2} = \overline{v^2} \int_0^t \int_0^t R_h(\alpha_2 - \alpha_1) d\alpha_1 \, d\alpha_2.$$

Taking into account that $R_h(h)$ is an even function

$$\overline{y^2} = \overline{v^2} \int_0^t d\alpha_1 \int_{-\alpha_1}^{t-\alpha_1} R_h(\alpha) d\alpha = \overline{v^2} [\int_0^t d\alpha_1 \int_0^{\alpha_1} R_h(\alpha) d\alpha + \int_0^t d\alpha_1 \int_0^{t-\alpha_1} R_h(\alpha) d\alpha].$$

and after inverting the order of integration and determining the new limits of integration, we find

$$\overline{y^2} = 2\overline{v^2} \int_0^t (t - \alpha) R_h(\alpha) d\alpha. \tag{1}$$

This relation was found by a different method by J. Kampé de Fériet[5] and is simpler for applications than the equation

$$\overline{y^2} = 2\overline{v^2} \int_0^t d\alpha_2 \int_0^{\alpha_2} R_h(\alpha_1) d\alpha_1 \tag{2}$$

originally demonstrated by G. I. Taylor.[1]

We shall consider two special cases: when the interval of time t is large and when it is small, as compared with the Lagrangian scale of turbulence $L_h = \int_0^\infty R_h(\alpha) d\alpha$.

Large Dispersion Time: When $t \gg L_h$ equation (1) becomes

$$(t \gg L_h) \qquad \overline{y^2} = 2\overline{v^2} L_h t - 2\overline{v^2} \int_0^\infty \alpha R_h(\alpha) d\alpha, \tag{3}$$

where the last term is constant. For very large dispersion times t, this constant becomes small as compared with the first term on the right and[1]

$$(t \gg L_h) \qquad \overline{y^2} = 2\overline{v^2} L_h t. \tag{4}$$

Small Dispersion Time: Consider now the case when $t \ll L_h$. Expanding the correlation function in a power series, taking into account its evenness, and neglecting terms of h of an order higher than the second, we find $R_h(h) \approx 1 - [h^2/(2\lambda_h^2)]$ where λ_h is called the Lagrangian microscale of turbulence. Substituting $R_h(h)$ in (1), we obtain

$$(t \ll L_h) \qquad \overline{y^2} = \left(1 - \frac{1}{12} \frac{t^2}{\lambda_h^2}\right) \overline{v^2} t^2. \tag{5}$$

When the ratio t/λ_h is small, the second term in the brackets may be neglected and[1]

$$(t \ll L_h) \qquad \overline{y^2} = \overline{v^2} t^2. \tag{6}$$

General Case: When the dispersion time t cannot be considered small or large as compared with the Lagrangian scale of turbulence, the variation of $\overline{y^2}$ as a function of t depends on the shape of the correlation curve. When this shape is known it may be possible to represent the correlation curve by a correlation function. Various representative functions[6] can be used

for this purpose and the variation of $\overline{y^2}$ as a function of t can then be found by using (1).

3. Mean Concentration Distribution.—Let us consider a turbulent fluid at rest and let us assume that a large number of fluid particles are concentrated at an initial time zero at the origin of a set of rectangular axes. After an interval of time t these particles are dispersed in a cloud by the influence of the turbulent fluctuations. If the turbulence is homogeneous and isotropic, the cloud of particles is spherically symmetrical.

In the preceding paragraph we have given the variance $\overline{y^2}$ for such a cloud of particles as a function of the characteristics of turbulence. We shall now investigate the distribution of particles within a cloud. This distribution can be expressed by $\text{Prob}[x < \alpha < x + dx,\ y < \beta < y + dy,\ z < \gamma < z + dz] = \psi(x, y, z)dx\,dy\,dz$. The function $\psi(x, y, z)$ is a probability density function and $\int_{-\infty}^{+\infty}\int\int \psi(x, y, z)dx\,dy\,dz = 1$. We assume[7] a three-dimensional Gaussian distribution and, therefore, a probability density function of the form

$$\psi(x, y, z) = \frac{k^3}{\pi^{3/2}} \exp\left[-k^2(x^2 + y^2 + z^2)\right]. \tag{7}$$

When the variance $\overline{y^2}$ for this distribution is determined, we find $k^2 = 2\overline{y^2}$.

4. Instantaneous Point-Source of Dispersion in a Turbulent Fluid at Rest.—Let us consider the case of a turbulent fluid at rest. At an initial moment 0, the particles are concentrated at the origin of the axes. The number (or quantity) of particles emitted at the initial moment is denoted by Q_0. $\bar{s}_0(x, y, z, t)$ is the mean concentration of particles at a point x, y z, and after a dispersion time t. The mean concentration is measured by the number (or quantity) of particles per unit of volume. We assume that Q_0 is very large (or that the particles are divisible) and, therefore, at the initial time $\bar{s}_0(0, 0, 0, 0) = \infty$, $\bar{s}_0(x, y, z, 0) = 0$.

Applying (7), we find that after a dispersion time t, the number of particles found in a volume element $dxdydz$ is equal to $\bar{s}_0(x, y, z, t)dxdydz$ with

$$\bar{s}_0(x, y, z, t) = \frac{Q_0}{(2\pi\overline{y^2})^{3/2}} \exp\left[-\frac{1}{2\overline{y^2}}(x^2 + y^2 + z^2)\right]. \tag{8}$$

In this equation $\overline{y^2}$ is to be expressed as a function of $\overline{v^2}$, $R_h(h)$ and t by (1).

When we consider the dispersion of particles in a turbulent flow whose mean velocity $\bar{u} = U$ is parallel to the x-axis, then the coordinate x in (8) should be replaced by $(x - Ut)$.

5. Continuous Point-Source of Dispersion in a Fluid Flow.—Consider now, in a fluid flow, the dispersion from a continuous point-source emitting steadily Q particles per unit of time. The origin of a set of rectangular

axes is placed at the point-source with the x-axis parallel to the direction of the mean velocity $\bar{u} = U$. Due to the choice of axes, the components of the turbulent velocity are $u' = u - U$, $v' = v$, $w' = w$. The mean concentration distribution for the time interval $(t, t + dt_0)$ of those Qdt_0 particles which were emitted during the interval of time $(t_0, t_0 + dt_0)$ can be represented by (8), if we replace $\bar{s}_0(x, y, z, t)$, Q_0 and x, respectively, by $\bar{S}(x, y, z, t - t_0)dt_0$, Qdt_0 and $(x - Ut)$. Taking the sum of the effects of the emission from $-\infty$ till t, we find that

$$\bar{s}_U(x, y, z) = \int_{-\infty}^{t} \bar{S}(x, y, z, t - \alpha)d\alpha = \int_0^{\infty} \bar{S}(x, y, z, \beta)d\beta \quad (9)$$

is independent of the dispersion time. $\bar{s}_U(x, y, z)U\,dydz$ is the average number of particles which pass per unit of time through an element of area $dydz$ of a plane perpendicular to the direction of the mean velocity U. From (8) and (9), we find the mean concentration flux

$$\bar{s}_U(x, y, z)U = U\int_0^{\infty} \frac{Q}{(2\pi\overline{y^2})^{3/2}} \exp\left\{-\frac{1}{2\overline{y^2}}[(x - U\beta)^2 + y^2 + z^2]\right\} d\beta \quad (10)$$

where $\overline{y^2} = 2v^2 \int_0^{\beta} (\beta - \gamma)R_h(\gamma)d\gamma$.

Non-dimensional Mean Concentration Flux: We shall now represent (10) in a non-dimensional form. With this representation it will be easier to examine the dependence of the mean concentration flux on the various parameters.

We introduce new coordinates $\xi = x/UL_h$, $\eta = y/UL_h$, $\zeta = z/UL_h$. The Lagrangian correlation function is represented as a function of h/L_h by $\Re_h(h/L_h) = R_h(h)$, and the intensity of turbulence is given by $T = \sqrt{\overline{u'^2}}/U = \sqrt{\overline{v'^2}}/U = \sqrt{\overline{w'^2}}/U$.

The mean concentration flux is represented as a function of the above coordinates by $\sigma_U(\xi, \eta, \zeta) = U^2L_h^2[\bar{s}_U(x, y, z)U/Q]$, and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma_U(\xi, \eta, \zeta)d\eta\,d\zeta = 1$. With the non-dimensional notations, (10) becomes

$$\sigma_U(\xi, \eta, \zeta) = \frac{1}{(2\pi)^{3/2}T^3}\int_0^{\infty} \frac{1}{\iota^3(\beta_1)} \exp\left\{-\frac{1}{2T^2\iota^2(\beta_1)} \times [(\xi - \beta_1)^2 + \eta^2 + \zeta^2]\right\} d\beta_1 \quad (11)$$

where $\iota^2(\beta_1) = 2\int_0^{\beta_1}(\beta_1 - \gamma_1)\Re_h(\gamma_1)d\gamma_1$ represents $\iota^2(\tau)$ for $\tau = t/L_h = \beta_1$.

We see immediately that the mean concentration distribution is a function of (in addition to the coordinates): (a) the shape of the Lagrangian correlation curve determined by the function $\Re_h(h/L_h)$; (b) the intensity of turbulence T and (c) the product UL_h of the mean velocity by the Lagrangian scale of turbulence.

We shall consider two cases: when the distance from the emission source

x is large and when it is small compared to UL_h, i.e., when ξ is, respectively, large and small compared to unity.

Large Distances from the Source: When the dispersion time is large, (4) can be applied and $\iota^2 = 2\tau$. Replacing $\iota^2(\beta_1)$ in (11) by $2\beta_1$, we find after integration

$$\sigma_U(\xi, \eta, \zeta) = \frac{1}{4\pi T^2 r} \exp\left(\frac{\xi - r}{2T^2}\right) \tag{12}$$

with $r = \sqrt{\xi^2 + \eta^2 + \zeta^2}$.

The dispersion factor ι was determined neglecting the diffusion in the direction of the mean flow. The value of ι corresponds, therefore, to a one-dimensional case of diffusion. We now recompute ι taking into account the three-dimensionality of the dispersion wake. The new dispersion factor ι_d represents the standard-deviation $\sqrt{y_d^2}$ of the mean concentration measured in a plane perpendicular to the direction of the mean velocity along a diameter passing through the dispersion axis $0x$. We have

$$\iota_d^2 = \frac{1}{L_h^2}\frac{\overline{y_d^2}}{\overline{v^2}} = \frac{\int_0^\infty \eta^2 \sigma_U(\xi, \eta, 0)d\eta}{T^2 \int_0^\infty \sigma_U(\xi, \eta, 0)d\eta}$$

Applying (12) and replacing in the two integrals the ratio η/ξ by a hyperbolic sine and using the Schläfli transformation, we find[7]

$$\iota_d^2 = 2\xi \frac{K_1(\xi/2T^2)}{K_0(\xi/2T^2)} \tag{13}$$

Since the ratio of the first order Bessel function K_1 to the zero order Bessel function K_0 tends to one when their common argument tends to infinity, we have

$$\lim_{\xi/2T^2 \to \infty} \iota_d^2 = 2\xi. \tag{14}$$

At the limit, we have, therefore, a relation corresponding to the equation which G. I. Taylor has given[2] for the one-dimensional case.

It should be noted that equation (13) is valid only when ξ is much larger than unity. On the other hand, when $\xi/2T^2$ is very large, equation (13) is well represented by its asymptotic relation (14). Equation (13) will, therefore, be most useful when at the same time both ξ and T are large. More particularly, equation (13) should be used when the turbulent velocities are of the same (or of a larger) order of magnitude as the mean velocity.

An approximate relation can be given for $\sigma_U(\xi, \eta, \zeta)$ for the case when η and ζ are small compared with ξ. Developing $(\xi - r)$ in a series and

neglecting the high order terms, we find[8] from (12)

$$\sigma_U(\xi, \eta, \zeta) \approx \frac{1}{4\pi T^2}\frac{1}{\xi}\exp\left(-\frac{1}{4T^2}\frac{\eta^2+\zeta^2}{\xi^2}\right). \qquad (15)$$

Small Distances from the Source: When the distance from the source is sufficiently small, we find from (6) that $\iota = \tau$. After replacing in (11) $\iota(\beta_1)$ by β_1, and performing the integration, we find[7]

$$\sigma_U(\xi, \eta, \zeta) = \frac{1}{(2\pi)^{3/2}}\frac{1}{r^2 T}\exp\left(-\frac{1}{2T^2}\right)\left[1+\sqrt{\frac{\pi}{2}}\frac{\xi}{Tr}\times \exp\left(\frac{1}{2T^2}\frac{\xi^2}{r^2}\right)\operatorname{erfc}\left(-\frac{1}{\sqrt{2}T}\frac{\xi}{r}\right)\right] \qquad (16)$$

where $r^2 = \xi^2 + \eta^2 + \zeta^2$ and $\operatorname{erfc}(X) = 1 - (2/\sqrt{\pi})\int_0^X \exp(-\alpha^2)d\alpha$.

When the intensity of turbulence T is small compared to (ξ/r), the second term in the brackets is large compared with the first term and neglecting the first, we find

$$\sigma_U(\xi, \eta, \zeta) \approx \frac{1}{4\pi}\frac{\xi}{T^2}\frac{1}{r^2}\exp\left[-\frac{1}{2T^2}\left(1-\frac{\xi^2}{r^2}\right)\right]\operatorname{erfc}\left(-\frac{1}{\sqrt{2}T}\frac{\xi}{r}\right). \qquad (17)$$

For small values of η and ζ as compared with ξ (and when T is small), we find the simple relation

$$\sigma_U(\xi, \eta, \zeta) \approx \frac{1}{2\pi T^2}\frac{1}{\xi^2}\exp\left(-\frac{1}{2T^2}\frac{\eta^2+\zeta^2}{\xi^2}\right). \qquad (18)$$

It should be noticed that only with equation (16) do we find finite mean concentrations upstream of the dispersion origin.

* This work was done under Contract NOrd 7386 with the U. S. Navy Bureau of Ordnance.

[1] Taylor, G. I., *Proc. Math. Soc.* (*London*), **20,** 196 (1921).

[2] Taylor, G. I., *Proc. Royal Soc.* (*London*), **151-A,** 421 (1935); Dryden, H. L., *Ind. Eng. Chem.*, **31,** 416 (1939).

[3] A random function depends on a parameter ω chosen at random in a measure space Ω with a measure equal to 1. The mean value of a function $\Psi(x, y, z, t, \omega)$ is then given by $\Psi(x, y, z, t) = \int_\Omega \Psi(x, y, z, t, \mu)d\mu$. In physical applications it is not possible to measure such mean values and time or space averages are used instead.

[4] The only correlation coefficient used here is the Lagrangian time correlation coefficient, found when the turbulent velocity of the same fluid particle is considered as a function of time.

[5] Kampé de Fériet, J., *Ann. Soc. Sci.* (*Bruxelles*), **59-I,** 145 (1939).

[6] Various representative correlation functions of the form $\exp[-k(|h|)^n]\varphi(h)$ with $n = 1$ and 2 are discussed in a paper prepared by the author in 1942 and published as *O.N.E.R.A.* (*Paris*), *Rep. Tech.* **34** (1948).

[7] Frenkiel, F. N., *U. S. Naval Ordnance Lab., NOLR.*, **1136,** 67 (1949).

[8] Roberts, O. F. T., has given a similar relation [cf. p. 36 of Sutton, O. G., *Atmos-*

pheric Turbulence, Methuen (London) (1949)]. To find his relation Roberts uses the differential equation of turbulent diffusion $(\partial\bar{s}_U/\partial t) + U(\partial\bar{s}_U/\partial x) = K\nabla^2\bar{s}_U$. It can be shown that the coefficient of eddy diffusion K is equal to $\overline{v'^2}L_h$, if the molecular diffusion is neglected. The differential equation is valid only when $t \gg L_h$. It is, however, possible [Frenkiel, F. N., *Proc. Micrometeorological Symposium, International Union Geodesy and Geophysics*, Brussels, Belgium, August, 1951] to define an "apparent coefficient of eddy diffusion" $K_t = (\overline{v'^2}/t)\int_0^t (t-\alpha)R_h(\alpha)\,d\alpha$, depending on the dispersion time t, which, when used instead of K in the above differential equation, makes the equation valid for all values of t.

THE NUMBER OF SOLUTIONS OF CERTAIN EQUATIONS IN A FINITE FIELD

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Communicated by E. T. Bell, April 18, 1952

Exact formulas for the number of solutions of certain special types of equations in a finite field have been obtained in recent papers by Hua and Vandiver,[1] Faircloth,[2] Faircloth and Vandiver;[3] see also Ward.[4] In the present note we state some additional theorems of this kind.

1. Let $a_1, \ldots, a_k$ be rational integers such that $(a_1, \ldots, a_k) = 1$; then we can construct a matrix of integers (a_{ij}), $i, j = 1, \ldots, k$, such that $a_{1j} = a_j$ and $|a_{ij}| = 1$. Now if F is an arbitrary field, consider the equation

$$\xi_1^{a_1}\ldots\xi_k^{a_k} = \alpha \qquad (\alpha, \xi_i \in F) \tag{1.1}$$

for fixed α. For $\alpha = 0$ the solution is evident; for $\alpha \neq 0$, no $\xi_i = 0$. We put

$$\eta_i = \prod_{j=1}^{k} \xi_j^{a_{ij}} \qquad (i = 1, \ldots, k),$$

which is equivalent to

$$\xi_i = \prod_{j=1}^{k} \eta_j^{b_{ij}} \qquad ((b_{ij}) = (a_{ij})^{-1}).$$

Thus (1.1) is equivalent to $\eta_1 = \alpha$, η_j arbitrary but $\neq 0$ for $j = 2, \ldots, k$.

2. Now let F be the finite field $GF(q)$, $q = p^n$, and consider the equation

$$\xi_1^{a_1}\ldots\xi_k^{a_k} = f(\zeta_1, \ldots, \zeta_r) \qquad (\xi_i, \zeta_j \in GF(q)), \tag{2.1}$$

where f denotes an arbitrary polynomial with coefficients in $GF(q)$. Applying the result of the previous paragraph we get

THEOREM 1. *Let N denote the number of solutions of (2.1) in ξ_i, ζ_j; let $A(\alpha)$ denote the number of solutions of*

$$f(\zeta_1, \ldots, \zeta_r) = \alpha \qquad (\alpha \in GF(q)); \tag{2.2}$$

then

$$N = (q^k - (q-1)^k)A(0) + (q-1)^{k-1} \sum_{\alpha \neq 0} A(\alpha). \tag{2.3}$$

When an explicit formula for the number of solutions of (2.2) is available then (2.3) will furnish an explicit formula for the number of solutions of (2.1). In particular using the well-known results concerning

$$Q(\zeta_1, \ldots, \zeta_r) = \alpha, \tag{2.4}$$

where Q denotes a quadratic form with coefficients in $GF(q)$ and with discriminant $\delta \neq 0$, we obtain

THEOREM 2. *The number of solutions of*

$$\xi_1^{a_1} \ldots \xi_k^{a_k} = Q(\zeta_1, \ldots, \zeta_r) \qquad (a_i \geq 1)$$

for $r = 2s$ is

$$q^{k+2s-1} + \lambda(q^k(q^s - q^{s-1}) - (q-1)^k q^s)$$

where $\lambda = +1$ or -1 according as $(-1)^s\delta$ is a square or a non-square of the field; for $r = 2s+1$ the number of solutions is q^{k+2s} or $q^{k+2s} + \lambda q^s(q-1)^k$ according as $(a_1, \ldots, a_k)$ is odd or even.

More generally we can determine the number of solutions of

$$\xi_1^{a_1} \ldots \xi_k^{a_k} = Q(U_1, \ldots, U_r) \qquad (a_i \geq 1), \tag{2.5}$$

in ξ_i, U_j where the U_j are polynomials $\in GF[q, x]$ of degree $< m$; for brevity we consider only the case r even. Using a result of Eckford Cohen[5] we obtain

THEOREM 3. *The number of solutions of (2.5), with $r = 2s$, is*

$$(q^k - (q-1)^k)\Big\{\lambda^m q^{ms} + (q^s - \lambda)q^{(s-1)(2m-1)} \sum_{z=0}^{m-1} \lambda^z q^{-z(z-2)}\Big\} + (q-1)^k(q^s - \lambda)q^{(s-1)(2m-1)},$$

where λ has the same meaning as in Theorem 2.

Similar results can be obtained if one or more of the U_j in (2.5) are restricted to be primary of degree m.

3. We consider next the equation

$$\sum_{i=1}^{t} \alpha_i \prod_{j=1}^{k_i} \xi_{ij}^{a_{ij}} = \alpha \qquad (\alpha, \alpha_i, \xi_{ij} \in GF(q)), \tag{3.1}$$

where

$$(a_{i1}, \ldots, a_{ik_i}) = 1 \qquad (i = 1, \ldots, t). \tag{3.2}$$

We have

THEOREM 4. *The number of solutions ξ_{ij} of (3.1) is*

$$q^{k-1} + lq^{-1} \prod_{i=1}^{t} (q^{k_i} - q(q-1)^{k_i-1}), \tag{3.3}$$

where $k = k_1 + \ldots + k_t$ and $l = q - 1$ for $\alpha = 0$, $l = -1$ for $\alpha \neq 0$. In particular the number of solutions of

$$\alpha_1 \xi_1^{a_1} \eta_1^{b_1} + \ldots + \alpha_t \xi_t^{a_t} \eta_t^{b_t} = \alpha \qquad ((a_i, b_i) = 1) \tag{3.4}$$

is $q^{2t-1} + lq^{t-1}$.

Let $P(\xi)$ denote the left member of (3.1) and consider the equation

$$P(\xi) = f(\zeta_1, \ldots, \zeta_r), \tag{3.5}$$

with f as in (2.1). Then we have the following extension of Theorem 1.

THEOREM 5. *The number of solutions of (3.5) is*

$$R(0)A(0) + R(1) \sum_{\alpha \neq 0} A(\alpha),$$

where $R(\alpha)$ denotes the number of solutions of (3.1) and $A(\alpha)$ the number of solutions of (2.2).

The second part of Theorem 4 leads to

THEOREM 6. *The number of solutions of*

$$P(\xi, \eta) = Q(\zeta_1, \ldots, \zeta_r), \tag{3.6}$$

where $P(\xi, \eta)$ denotes the left member of (3.4) and Q denotes a quadratic form is given by

$$\begin{cases} q^{2s+2t-1} + \lambda(q-1)q^{s+t-1} & (r = 2s), \\ q^{2s+2t} & (r = 2s+1), \end{cases}$$

where λ has the same meaning as in Theorem 2.

We can also obtain a result like Theorem 3 if in (3.6) we replace ζ_j by U_j, polynomials in $GF[q, x]$ of degree $< m$. However, we shall omit this generalization and state instead the following

THEOREM 7. *Let f denote an arbitrary polynomial with coefficients $\epsilon GF(q)$ and let Q denote a quadratic form; then the number of solutions of*

$$f(\xi_1, \ldots, \xi_t) = Q(\zeta_1, \ldots, \zeta_{2s}) \tag{3.7}$$

is

$$A(q^{2s-1} + \lambda(q^s - q^{s-1})) + (q^t - A)(q^{2s-1} - \lambda q^{s-1}),$$

where A denotes the number of solutions of $f(\xi_1, \ldots, \xi_t) = 0$ and λ has its

usual meaning. In particular if the polynomial f never vanishes, then the number of solutions of (3.7) is

$$q^t(q^{2s-1} - \lambda q^{s-1}).$$

The corresponding result when Q is a form in $2s + 1$ unknowns is more complicated. In the following special case, however, a simple result is obtained.

THEOREM 8. *If f is a polynomial that never vanishes, then the number of solutions ξ_i, ζ_j of*

$$f^2(\xi_1, \ldots, \xi_j) = Q(\zeta_1, \ldots, \zeta_{2s+1})$$

is

$$q^t(q^{2s} + \mu q^s),$$

where $\mu = +1$ or -1 according as $(-1)^s\delta$ is or is not a square, and δ is the discriminant of Q.

4. In the next place we state a theorem in which the condition (3.2) is weakened somewhat. We shall require some notation. For $\beta \in GF(q)$ put

$$e(\beta) = e^{2\pi i t(\beta)/p}, \quad t(\beta) = \beta + \beta^p + \ldots + \beta^{p^{n-1}},$$

$$G(\beta) = \sum_{\xi} e(\beta\xi^2) \qquad (\beta \neq 0).$$

Also let $\psi(\beta) = +1, -1, 0$ according as β is a square, a non-square or zero in $GF(q)$. It follows that

$$G(\beta) = \psi(\beta)G(1), \quad G^2(1) = q\psi(-1). \tag{4.1}$$

We now consider the equation

$$\alpha_1\xi_1^{a_1}\eta_1^{b_1} + \ldots + \alpha_r\xi_r^{a_r}\eta_r^{b_r} = \alpha, \tag{4.2}$$

where $(a_i, b_i) = 2$ for $i = 1, \ldots, r$.

THEOREM 9. *The number of solutions ξ_i, η_i of (4.2) is*

$$q^{r-1} + q^{-1} \sum_{\beta \neq 0} e(-\alpha\beta) \prod_{i=1}^{r} (q + (q-1)\psi(\alpha_i\beta)G(1)). \tag{4.3}$$

By means of (4.1), (4.3) can be simplified; however, we shall not take the space to write down the final formulas. We remark that in place of (4.2) we can consider an equation of the form (3.1) with $(a_{i1}, \ldots, a_{ik_i}) = 1$ or 2, but the final formulas for the number of solutions are rather complicated.

5. Theorem II of reference 1 suggests the following additional result.

THEOREM 10. *Let $(a_{i1}, \ldots, a_{ik_i}) = d_i$, $(d_i, d_j) = 1$ for $i \neq j$. Then the number of solutions of*

$$\sum_{i=1}^{t} \alpha_i \prod_{j=1}^{k_i} \xi_{ij}^{a_{ij}} = 0 \qquad (5.1)$$

with $\prod_{i,j} \xi_{ij} \neq 0$, *is*

$$q^{k-1}(q-1)((q-1)^{t-1} + (-1)^t) \qquad (k = k_1 + \ldots + k_t).$$

The total number of solutions of (5.1) *is* q^{k+t-1}, *where* $k = k_1 + \ldots + k_t$.

[1] These PROCEEDINGS, **35,** 481–487 (1949).
[2] *Ibid.*, **37,** 619–622 (1951).
[3] *Ibid.*, **38,** 52–57 (1952).
[4] *Ibid.*, **37,** 113–114 (1951).
[5] *Duke Math. J.*, **14,** 543–557 (1947).

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*ON SPECTRAL ANALYSIS OF STATIONARY TIME SERIES**

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Communicated by Saunders MacLane, April 30, 1952

1. The present statistical theory of analysis of stationary time series (e.g., extrapolation) has assumed complete knowledge of the covariance sequence or equivalently of the spectrum of the process. It is, therefore, of great importance to be able to estimate one of these. However, knowledge of the spectrum seems to yield greater immediate insight into the structure of the process. This seems to have first been noted in a fundamental paper by Bartlett.[1] An unpublished paper by Tukey[2] deals with some aspects of the problem of estimating the spectrum.

2. Consider a discrete parameter stochastic process x_t ($t = \ldots, -1, 0, 1, \ldots$) of the following form:

$$x_t = \sum_{\nu=-\infty}^{\infty} a_{t-\nu}\xi_\nu$$

where $\{a_n\}$ is a real sequence and

$$\sum_{-\infty}^{\infty} a_n^2 < \infty.$$

The ξ_ν's are assumed to be independent and identically distributed real-valued random variables with

$$E\xi_\nu = 0, \; E\xi_\nu^2 = \sigma^2, \; E\xi_\nu^4 = \mu_4.$$

The conditions cited above imply that the stochastic process x_t is strictly

stationary. Such a process is known to have an absolutely continuous spectral distribution function $F(\lambda)$ with a density given by

$$F'(\lambda) = f(\lambda) = \frac{1}{2\pi}\left|\sum_{-\infty}^{\infty} a_\nu e^{i\nu\lambda}\right|^2 \sigma^2.$$

Proofs of the results cited below will require $f(\lambda)$ to satisfy a smoothness condition and the ξ_ν's to have moments up to the eighth order.

As the process is real-valued, the spectral density is symmetric about zero and hence one need only consider a spectral distribution function $F(\lambda)$ as the integral of the spectral density from zero to λ, $0 \leq \lambda \leq \pi$. A reasonable estimate of the spectral distribution function $F(\lambda)$ is

$$F_N^*(\lambda) = \int_0^\lambda I_N(l)dl$$

where

$$I_N(l) = \frac{1}{2\pi N}\left|\sum_1^N x_n e^{inl}\right|^2$$

is the so-called periodogram. Professor Doob suggested that we investigate the problem of giving confidence bands for the estimation of $F(\lambda)$ in terms of $F_N^*(\lambda)$. In the case of a normally distributed process, Theorem 3 yields a solution which is non-parametric with respect to the spectrum. In the general case, Theorem 1 and an appropriate estimate of $e = \frac{\mu_4}{\sigma^4} - 3$ can be used to obtain conservative bands non-parametric with respect to the spectrum. Theorem 4 can be used to test whether two independently drawn time series come from processes having the same spectrum.

3. The theorems are as follows:

THEOREM 1.

$$\lim_{N\to\infty} P\left\{\max_{0\leq\lambda\leq\pi}\left|N[F_N^*(\lambda) - F(\lambda)]\right| \leq \alpha\right\} = P\left\{\max_{0\leq\lambda\leq\pi}\left|Z(\lambda)\right| \leq \alpha\right\}$$

where $Z(\lambda)$ is a normal process whose distribution is completely determined by the mean value $EZ(\lambda) = 0$ and the covariance

$$EZ(\lambda)Z(\mu) = eF(\lambda)F(\mu) + 2\pi G(\text{min. } (\lambda, \mu)),$$

where

$$G(\lambda) = \int_0^\lambda f^2(l)dl.$$

COROLLARY. *If $e = 0$*

$$\lim_{N\to\infty} P\left\{\max_{0\leq\lambda\leq\pi}\left|\sqrt{N}[F_N^*(\lambda) - F(\lambda)]\right| \leq \alpha\right\} = P\left\{\max_{0\leq\lambda\leq 2\pi G(\pi)}\left|W(\lambda)\right| \leq \alpha\right\}$$

where $W(\lambda)$ is the Wiener process, i.e.,

$$EW(\lambda) = 0 \qquad EW(\lambda)W(\mu) = \text{min. } (\lambda, \mu)$$

and $W(\lambda)$ *is normally distributed.*

Note that $e = 0$ when the ξ_ν are normally distributed.

THEOREM 2. *The statistic*

$$R_N = \frac{1}{2} \sum_{\nu=-kN^\alpha}^{kN^\alpha} \frac{c_\nu^2}{N^2}$$

converges to $2\pi G(\pi)$ *in probability as* $N \to \infty$ *if* $k > 0, 0 < \alpha < 1$. *The* c_ν*'s are the product sums*

$$c_\nu = \sum_{1}^{N-|\nu|} x_n x_{n+|\nu|}.$$

THEOREM 3. *If* $e = 0$

$$\lim_{N\to\infty} P\left\{ \frac{\max_{0\le\lambda\le\pi} \left|\sqrt{N}[F_N^*(\lambda) - F(\lambda)]\right|}{R_N^{1/2}} \le \alpha \right\} = P\left\{ \max_{0\le\lambda\le 1} |W(\lambda)| \le \alpha \right\}.$$

It is well known that the probability on the right can be explicitly evaluated in a convenient form.

THEOREM 4. *If* $e = 0$ *and if* $F_{N1}^*(\lambda)$, $F_{N2}^*(\lambda)$, R_{N1}, R_{N2} *are the statistics defined above corresponding to two independently drawn samples from the same stochastic process, then*

$$\lim_{N\to\infty} P\left\{ \frac{\max_{0\le\lambda\le\pi} \left|\sqrt{N}[F_{N1}^*(\lambda) - F_{N2}^*(\lambda)]\right|}{(R_{N1} + R_{N2})^{1/2}} \le \alpha \right\} = P\left\{ \max_{0<\lambda\le\pi} |W(\lambda)| \le \alpha \right\}.$$

4. Although the plausibility of the basic Theorem 1 is apparent by heuristic arguments, the proof requires a long and formidable sequence of lemmas. The practical value of the theory cited above can only be judged after having completed an extensive computational program involving analyses of artificially generated time series. Some preliminary analyses have already been made and tables of the related distributions are being compiled. Considerable investigation must be made of the practical application of the theory. The proofs of the above theorems and considerable elaboration of the theory (e.g., solution of the corresponding problem for the spectral density) as well as aspects of the practical application of the theory will be contained in a forthcoming publication.

* This work was supported by a grant from the Office of Naval Research.

[1] Bartlett, M. S., "Periodogram Analysis and Continuous Spectra," *Biometrika*, **37**, (1950).

[2] Tukey, J. W., "Measuring Noise Color," unpublished manuscript.

ON THE THEOREM OF RIEMANN-ROCH FOR ADJOINT SYSTEMS ON KÄHLERIAN VARIETIES

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Communicated by Hermann Weyl, March 10, 1952

1. Let $\mathfrak{M}$ be a compact Kählerian variety of (complex) dimension m. Suppose a (possibly reducible) compact analytic subvariety S of $\mathfrak{M}$ of dimension $m-1$ as given and denote by $\mathfrak{W}(S;\mathfrak{M})$ the linear space consisting of all meromorphic m-ple differentials W on $\mathfrak{M}$ which are *multiples* of $-S$ in the sense that $(W)+S\geqq 0$, (W) being the *divisor* of W. The problem concerning the *theorem of Riemann-Roch for the adjoint system*[1] of S consists in expressing the dimension of the linear space $\mathfrak{W}(S;\mathfrak{M})$ in terms of "numerical characteristics" of $\mathfrak{M}$ and S. The purpose of the present short note is to give an answer to this problem[2] in the following two simple cases: (*a*) *S is an irreducible non-singular variety*, (*b*) *S consists of two irreducible components S_1, S_2 which are both non-singular and the intersection $\Gamma=S_1\cap S_2$ is an irreducible non-singular variety having the intersection-multiplicity* 1 in the sense that at each point $\mathfrak{q}\in\Gamma$ the two varieties S_1, S_2 have different tangential planes.

2. We start with the following remark: Let V be a compact analytic non-singular subvariety of $\mathfrak{M}$ of (complex) dimension $m-h$ and $\mathfrak{t}$ be an arbitrary pure p-current[3] of type s on V. Then we can associate with $\mathfrak{t}$ a pure $(p+2h)$-current $T(\mathfrak{t})$ of type $(s+h)$ by setting $T(\mathfrak{t})[\Phi]=\mathfrak{t}[\Phi_V]$ for a "variable" $(2m-p-2h)$-form Φ of Class C^∞ on $\mathfrak{M}$, where Φ_V *denotes the $(2m-p-2h)$-form on V induced by Φ*. Clearly the correspondence $\mathfrak{t}\to T(\mathfrak{t})$ is linear and one-to-one. Moreover we have $dT(\mathfrak{t})=T(d\mathfrak{t})$, since $(d\Phi)_V=d(\Phi_V)$. In view of these facts, *we identify* $\mathfrak{t}$ *with* $T(\mathfrak{t})$ *and consider* $\mathfrak{t}=T(\mathfrak{t})$ *as a pure $(p+2h)$-current of type $s+h$ on $\mathfrak{M}$.*

3. First we consider *the second case* (*b*). Every m-ple differential $W\in\mathfrak{W}(S;\mathfrak{M})$ can be considered as a pure m-current of type 0 on $\mathfrak{M}$, since the integral $W[\Psi]=\int_{\mathfrak{M}}W\cdot\Psi$ converges absolutely for an arbitrary m-form Ψ of Class C^∞ on $\mathfrak{M}$. Similarly, every $(m-1)$-ple differential $\omega_\lambda\in\mathfrak{W}(\Gamma;S_\lambda)$ is a pure $(m-1)$-current of type 0 on S_λ ($\lambda=1$ or 2) and consequently *ω_λ can be considered as a pure $(m+1)$-current of type* 1 *on $\mathfrak{M}$. The exterior derivative dW of the current W has the representation $dW=\omega_1+\omega_2$, where $\omega_\lambda\in\mathfrak{W}(\Gamma;S_\lambda)$* ($\lambda=1,2$). This can be proved as follows: First, let $\mathfrak{q}$ be a point on $\Gamma=S_1\cap S_2$ and $(z_{\mathfrak{q}1},z_{\mathfrak{q}2},\ldots,z_{\mathfrak{q}m})$ be a system of local coordinates on $\mathfrak{M}$ with the center $\mathfrak{q}$ such that S_1 or S_2 coincides in a neighborhood $\mathfrak{U}(\mathfrak{q})$ of $\mathfrak{q}$ with the coordinate plane $z_{\mathfrak{q}1}=0$ or $z_{\mathfrak{q}2}=0$, respectively. Then W has in $\mathfrak{U}(\mathfrak{q})$ the expression $W=(z_{\mathfrak{q}1}z_{\mathfrak{q}2})^{-1}$

$F_q dz_{q1} dz_{q2} \ldots dz_{qm}$, where F_q is a holomorphic function defined in $\mathfrak{U}(q)$. By a simple calculation, we get

$$dW(\Phi) = 2\pi i \{ \int_{S_1} z_2^{-1} F_q dz_2 dz_3 \ldots dz_m \cdot \Phi_{S_1} - \int_{S_2} z_1^{-1} F_q dz_1 dz_3 \ldots dz_m \cdot \Phi_{S_2} \}$$

for an arbitrary $(m-1)$-form Φ of Class C^∞ whose carrier lies in $\mathfrak{U}(q)$, where $z_k = z_{qk}$ $(k = 1, 2, \ldots, m)$. Consequently, setting

$$\omega_{1q} = 2\pi i\, z_{q2}^{-1} F_q(0, z_{q3}, z_{q3}, \ldots, z_{qm}) dz_{q2} dz_{q3} \ldots dz_{qm},$$

$$\omega_{2q} = 2\pi i\, z_{q1}^{-1} F_q(z_{q1}, 0, z_{q3}, \ldots, z_{qm}) dz_{q1} dz_{q3} \ldots dz_{qm}$$

and considering the $(m-1)$-ple differentials $\omega_{\lambda q}$ on $S_\lambda \cap \mathfrak{U}(q)$ $(\lambda = 1, 2)$ thus defined as pure $(m+1)$-currents of Type 1 in $\mathfrak{U}(q)$, we obtain

$$dW = \omega_{1q} + \omega_{2q}, \quad \text{in } \mathfrak{U}(q) \qquad (q \in \Gamma). \tag{1}$$

Second, take a point $p \in S_\lambda - \Gamma$ ($\lambda = 1$ or 2) and choose a system of local coordinates $(z_{p1}, z_{p2}, \ldots, z_{pm})$ with the center p such that S_λ coincides in a neighborhood $\mathfrak{U}(p)$ of p with the coordinate plane $z_{p1} = 0$. Then W has in $\mathfrak{U}(p)$ the expression $W = z_{p1}^{-1} F_p dz_{p1} dz_{p2} \ldots dz_{pm}$, where F_p is a holomorphic function defined in $\mathfrak{U}(p)$. Moreover, setting $\omega_{\lambda p} = 2\pi i\, F_p(0, z_{p2}, \ldots z_{pm}) \cdot dz_{p2} \ldots dz_{pm}$, we get, in the same manner as above,

$$dW = \omega_{\lambda p}, \quad \text{in } \mathfrak{U}(p) \quad (p \in S_\lambda - \Gamma). \tag{2}$$

It follows from (1) and (2) that dW has the expression $dW = \omega_1 + \omega_2$, $\omega_\lambda \in \mathfrak{W}(\Gamma; S_\lambda)$.

Now we shall show that, *for given differentials* $\omega_\lambda \in \mathfrak{W}(\Gamma; S_\lambda)$ $(\lambda = 1, 2)$, *there exists a differential* $W \in \mathfrak{W}(S; \mathfrak{M})$ *with* $dW = \omega_1 + \omega_2$ *if and only if* ω_1, ω_2 *satisfy* $d\omega_1 + d\omega_2 = 0$ *and* $H\omega_1 + H\omega_2 = 0$ *as* $(m+1)$*-currents on* $\mathfrak{M}$, *where* $H\omega_\lambda$ *denotes the harmonic part*[4] *of the current* ω_λ. It is obvious that the existence of W with $dW = \omega_1 + \omega_2$ implies $d(\omega_1 + \omega_2) = 0$ and $H(\omega_1 + \omega_2) = 0$. Assume conversely that $d(\omega_1 + \omega_2) = H(\omega_1 + \omega_2) = 0$. Then with the help of Green's operator G and the Hodge-Weil's operator Λ, we can construct an m-current[5] $W = (-id\Lambda + \delta)G(\omega_1 + \omega_2)$ of Type 0 satisfying $dW = \omega_1 + \omega_2$. Moreover this current W is a holomorphic m-ple differential in $\mathfrak{M} - S$. In order to show that W is a meromorphic differential belonging to $\mathfrak{W}(S; \mathfrak{M})$, let p, z_{pk}, q, z_{qk} have the same meaning as above. Then ω_λ has on $S_\lambda \cap \mathfrak{U}(p)$ the expression $\omega_\lambda = f_p(z_{p2}, \ldots, z_m) dz_{p2} \ldots dz_{pm}$, where f_p is a holomorphic function defined on $S_\lambda \cap \mathfrak{U}(p)$. Setting $\theta_p = (2\pi i)^{-1} z_{p1}^{-1} f_p(z_{p2}, \ldots, z_{pm}) dz_{p1} dz_{p2} \ldots dz_{pm}$, we get therefore a pure m-current θ_p of Type 0 defined in $\mathfrak{U}(p)$ satisfying $d\theta_p = \omega_\lambda$. The difference $W - \theta_p$ satisfies $d(W - \theta_p) = 0$ and consequently $W - \theta_p$ must be a holomorphic differential[6] in $\mathfrak{U}(p)$. This proves that $z_{p1} W$ *is a holomorphic differential in* $\mathfrak{U}(p)$ *for each* $p \in S_\lambda - \Gamma$. For $q \in \Gamma$, it follows from the above result that $z_{1q} z_{2q} W$ is holomorphic in $\mathfrak{U}(q) - \Gamma$; hence, by virtue of a theorem of Hartogs,[7] $z_{1q} z_{2q} W$ *must be a holomorphic differential*

in $\mathfrak{U}(\mathfrak{q})$ *for each* $\mathfrak{q} \in \Gamma$. Thus we see that W is a meromorphic differential belonging to $\mathfrak{W}(S; \mathfrak{M})$.

4. *We denote by* $\mathfrak{G}_p(\mathfrak{M})$ *the linear space consisting of all p-ple differentials of the first kind on* $\mathfrak{M}$ *and set* $g_p(\mathfrak{M}) = \dim \mathfrak{G}_p(\mathfrak{M})$. In *the first case* (*a*), it can be shown in the same manner as above that, for every $W \in \mathfrak{W}(S; \mathfrak{M})$, $\omega = dW$ is an element of $\mathfrak{G}_{m-1}(S)$ and that, for given $\omega \in \mathfrak{G}_{m-1}(S)$, there exists $W \in \mathfrak{W}(S, \mathfrak{M})$ with $dW = \omega$ if and only if $H\omega = 0$, where each element of $\mathfrak{G}_{m-1}(S)$ is to be regarded as a pure $(m+1)$-current of Type 1 on $\mathfrak{M}$. Let $\{B_\nu | \nu = 1, 2, \ldots, g_{m-1}(\mathfrak{M})\}$ be a base of the linear space $\mathfrak{G}_{m-1}(\mathfrak{M})$ which is normalized in the sense that $\int_{\mathfrak{M}} B_\nu \cdot * \bar{B}_\mu = \delta_{\nu\mu}$. Then $H\omega$ has the expression $H\omega = (-1)^{m-1} \sum_\nu \omega[\bar{B}_\nu] * B_\nu$. Consequently, *there exists* $W \in \mathfrak{W}(S, \mathfrak{M})$ *with* $dW = \omega$ *if and only if* ω *satisfies*

$$\omega[B] = 0, \qquad \text{for all } B \in \mathfrak{G}_{m-1}(\mathfrak{M}), \tag{3}$$

where ω is a *given* element of $\mathfrak{G}_{m-1}(S)$. Thus, denoting by $\mathfrak{H}(S)$ the subspace of $\mathfrak{G}_{m-1}(S)$ consisting of all $\omega \in \mathfrak{G}_{m-1}(S)$ satisfying (3), we infer that the linear mapping $W \to \omega = dW$ maps $\mathfrak{W}(S, \mathfrak{M})$ *onto* $\mathfrak{H}(S)$, while dW vanishes if and only if W belongs to the subspace $\mathfrak{G}_m(\mathfrak{M})$ of $\mathfrak{W}(S, \mathfrak{M})$. Hence we get the relation $\dim \mathfrak{W}(S, \mathfrak{M}) = g_m(\mathfrak{M}) + \dim \mathfrak{H}(S)$. On the other hand, $\omega[\bar{B}] = \omega[\bar{B}_S] = (-1)^{(m-1)(m-2)/2}(i)^{m-1} \int_S \omega \cdot * \bar{B}_S$, and $B_S \in \mathfrak{G}_{m-1}(S)$ for all $B \in \mathfrak{G}_{m-1}(\mathfrak{M})$. Hence $\mathfrak{H}(S)$ is the subspace of $\mathfrak{G}_{m-1}(S)$ which is *orthogonal* to the subspace $\{B_S | B \in \mathfrak{G}_{m-1}(\mathfrak{M})\}$. Setting $l = \dim \{B | B \in \mathfrak{G}_{m-1}(\mathfrak{M}), B_S = 0\}$, we infer therefore that $\dim \mathfrak{H}(S) = g_{m-1}(S) - g_{m-1}(\mathfrak{M}) + l$. Combining this with the above result, we obtain the following

THEOREM 1. *In the first case* (*a*), *we have*

$$\dim \mathfrak{W}(S, \mathfrak{M}) = g_m(\mathfrak{M}) - g_{m-1}(\mathfrak{M}) + g_{m-1}(S) + l, \tag{4}$$

where l *is the number of linearly independent* $(m-1)$*-ple differentials* B *of the first kind on* $\mathfrak{M}$ *which vanish on* S *in the sense that* $B_S = 0$, B_S *being the differential induced on* S *by* B.

Remark: In case $\omega \in \mathfrak{G}_{m-1}(S)$ satisfies (3), the current $W = (-id\Lambda + \delta)G\omega$ satisfies not only $dW = \omega$ but $HW = 0$ and therefore $W[P] = 0$ for all $P \in \mathfrak{G}_m(\mathfrak{M})$.

5. We turn to the *second case* (*b*). Let $\mathfrak{w}^\circ$ be the subspace of $\mathfrak{W}(\Gamma; S_1) + \mathfrak{W}(\Gamma; S_2)$ consisting of all $\omega_1 + \omega_2$, $\omega_\lambda \in \mathfrak{W}(\Gamma; S_\lambda)$ $(\lambda = 1, 2)$, satisfying the conditions $d\omega_1 + d\omega_2 = 0$, $H\omega_1 + H\omega_2 = 0$. Then the result of Section 3 shows that the linear mapping $W \to \omega_1 + \omega_2 = dW$ maps $\mathfrak{W}(S; \mathfrak{M})$ *onto* $\mathfrak{w}^\circ$, while dW vanishes if and only if $W \in \mathfrak{G}_m(\mathfrak{M})$. Hence we get

$$\dim \mathfrak{W}(S; \mathfrak{M}) = g_m(\mathfrak{M}) + \dim \mathfrak{w}^\circ. \tag{5}$$

Clearly the condition $H\omega_1 + H\omega_2 = 0$ is equivalent to the following one:

$$\omega_1[\bar{B}] + \omega_2[\bar{B}] = 0, \qquad \text{for all } B \in \mathfrak{B}_{m-1}(\mathfrak{M}). \tag{6}$$

Now, applying the argument exponded in Section 4 to each pair of varieties S_λ, Γ ($\lambda = 1, 2$) instead of $\mathfrak{M}$, S, we infer that, for an arbitrary element $\omega_1 + \omega_2 \in \mathfrak{w}^\circ$, the exterior derivative $\xi = d\omega_1 = -d\omega_2$ belongs to $\mathfrak{B}_{m-2}(\Gamma)$ and satisfies as a current on S_λ the relation

$$\xi[\alpha_\lambda] = 0, \qquad \text{for all } \alpha_\lambda \in \mathfrak{B}_{m-2}(S_\lambda) \qquad (\lambda = 1, 2) \tag{7}$$

and that, if an element ξ of $\mathfrak{B}_{m-2}(\Gamma)$ satisfies (7), there exist $\omega_\lambda \in \mathfrak{W}(\Gamma; S_\lambda)$ ($\lambda = 1, 2$) satisfying $d\omega_1 = -d\omega_2 = \xi$ and

$$\omega_\lambda[\bar{\beta}_\lambda] = 0, \qquad \text{for all } \beta_\lambda \in \mathfrak{B}_{m-1}(S_\lambda)\ (\lambda = 1, 2) \tag{8}$$

(see the remark at the end of Section 4). Obviously (8) implies (6). Hence the linear mapping $\omega_1 + \omega_2 \to \xi = d\omega_1 = -d\omega_2$ maps $\mathfrak{w}^\circ$ *onto* the subspace $\mathfrak{H}(\Gamma)$ of $\mathfrak{B}_{m-2}(\Gamma)$ consisting of all $\xi \in \mathfrak{B}_{m-2}(\Gamma)$ satisfying (7). The kernel $\mathfrak{k}$ of this mapping is given by

$$\mathfrak{k} = \{\omega_1 + \omega_2 \,|\, \omega_\lambda \in \mathfrak{B}_{m-1}(S_\lambda),\ \omega_1[\bar{B}_{S_1}] + \omega_2[\bar{B}_{S_2}] = 0\ (B \in \mathfrak{B}_{m-1}(\mathfrak{M}))\}.$$

Setting $l = \dim \{B \,|\, B \in \mathfrak{B}_{m-1}(\mathfrak{M}),\ B_{S_1} = B_{S_2} = 0\}$, we get $\dim \mathfrak{k} = g_{m-1}(S_1) + g_{m-1}(S_2) - g_{m-1}(\mathfrak{M}) + l$ and therefore

$$\dim \mathfrak{w}^\circ = g_{m-1}(S_1) + g_{m-1}(S_2) - g_{m-1}(\mathfrak{M}) + l + \dim \mathfrak{H}(\Gamma). \tag{9}$$

It is obvious that the condition (7) is equivalent to

$$\xi[\alpha_{1\Gamma} - \alpha_{2\Gamma}] = 0, \qquad \text{for all } \alpha_\lambda \in \mathfrak{B}_{m-2}(S_\lambda) \qquad (\lambda = 1, 2). \tag{10}$$

Hence, setting $g_{m-2}(S) = \dim \{\alpha_1 + \alpha_2 \,|\, \alpha_\lambda \in \mathfrak{B}_{m-2}(S_\lambda),\ \alpha_{1\Gamma} = \alpha_{2\Gamma}\}$, we get

$$\dim \mathfrak{H}(\Gamma) = g_{m-2}(\Gamma) - g_{m-2}(S_1) - g_{m-2}(S_2) + g_{m-2}(S).$$

Combining this with (9) and inserting it in (5), we obtain the following

THEOREM 2. *In the second case (b), we have*

$$\begin{aligned}\dim \mathfrak{W}(S; \mathfrak{M}) = {} & g_m(\mathfrak{M}) - g_{m-1}(\mathfrak{M}) + g_{m-1}(S_1) - g_{m-2}(S_1) \\ & + g_{m-1}(S_2) - g_{m-2}(S_2) + g_{m-2}(\Gamma) + g_{m-2}(S) + l,\end{aligned} \tag{11}$$

where l is the number of linearly independent differentials $B \in \mathfrak{B}_{m-1}(\mathfrak{M})$ satisfying $B_{S_1} = B_{S_2} = 0$ and $g_{m-2}(S)$ is the dimension of the linear space $\{\alpha_1 + \alpha_2 \,|\, \alpha_\lambda \in \mathfrak{B}_{m-2}(S_\lambda)\ (\lambda = 1, 2),\ \alpha_{1\Gamma} = \alpha_{2\Gamma}\}$.

6. In this section we shall consider the case in which $\mathfrak{M}$ is an (irreducible non-singular) *algebraic variety* imbedded in a projective space. Let E be an arbitrary hyperplane section of $\mathfrak{M}$ which is irreducible and non-singular. Then a *theorem of Lefschetz*[8] asserts that, in case $p \leqq m - 1$ ($m = \dim \mathfrak{M}$), every p-cycle Z on $\mathfrak{M}$ is homologous to a p-cycle Z_E lying on E and that, in case $p \leqq m - 2$, a p-cycle Z_E lying on E is homologous to zero *on* E if Z_E is homologous to zero *on* $\mathfrak{M}$. It follows from this theorem that

(I) *in case* $p \leqq m-1$, *a* p*-ple differential* $P \in \mathfrak{S}_p(\mathfrak{M})$ *vanishes identically if* P_E *vanishes identically on* E *and that* (II) *in case* $p \leqq m-2$, *the mapping* $P \to P_E$ *maps* $\mathfrak{S}_p(\mathfrak{M})$ *isomorphically onto* $\mathfrak{S}_p(E)$. The second proposition (II) implies obviously that $g_p(E) = g_p(\mathfrak{M})$ for $p \leqq m-2$. Incidentally we denote by K the *canonical divisor* on $\mathfrak{M}$.

Now, for an arbitrary irreducible non-singular algebraic variety V of dimension n, we set

$$a(V) = g_n(V) - g_{n-1}(V) + g_{n-2}(V) - \ldots + (-1)^{n-1}g_1(V). \quad (12)^9$$

Then we have

THEOREM 3. *The dimension of the complete linear system* $|K+E|$ *on* $\mathfrak{M}$ *is given by*

$$\dim |K+E| = a(\mathfrak{M}) + a(E) - 1. \quad (13)$$

Proof: Since $\dim |K+E| = \dim \mathfrak{W}(E; \mathfrak{M}) - 1$, we get, by Theorem 1,

$$\dim |K+E| = g_m(\mathfrak{M}) - g_{m-1}(\mathfrak{M}) + g_{m-1}(E) + l - 1. \quad (14)$$

Now, it follows from the above proposition (I) that $l = 0$, while $g_p(E) = g_p(\mathfrak{M})$ for $p \leqq m-2$. Hence we obtain from (14) the formula (13), q. e. d.

THEOREM 4. *Let* S *be an arbitrary irreducible non-singular subvariety of* $\mathfrak{M}$ *of dimension* $m-1$ *and* E *be an irreducible non-singular hyperplane section of* $\mathfrak{M}$ *which cuts out on* S *an irreducible non-singular subvariety* ES *with the intersection multiplicity* 1. *Then the dimension of the complete linear system* $|K+S+E|$ *is given by*

$$\dim |K+S+E| = a(\mathfrak{M}) + a(S) + a(E) + a(ES) - 1. \quad (15)$$

Proof: By virtue of Theorem 2, we get

$$\begin{aligned}\dim |K+S+E| = {} & g_m(\mathfrak{M}) - g_{m-1}(\mathfrak{M}) + g_{m-1}(S) - g_{m-2}(S) \\ & + g_{m-1}(E) - g_{m-2}(E) + g_{m-2}(ES) + g_{m-2}(S+E) + l - 1.\end{aligned}$$

Now, we have $g_p(E) = g_p(\mathfrak{M})$ for $p \leqq m-2$ and, since ES is a hyperplane section of S, $g_p(ES) = g_p(S)$ for $p \leqq m-3$, while it follows from the proposition (I) that $l = 0$. Hence we obtain

$$\begin{aligned}\dim |K+S+E| = {} & a(\mathfrak{M}) + a(S) + a(E) + a(ES) - 1 + g_{m-2}(S+E) \\ & - g_{m-2}(\mathfrak{M}).\end{aligned}$$

It is sufficient therefore to show that $g_{m-2}(S+E) = g_{m-2}(\mathfrak{M})$. $g_{m-2}(S+E)$ is the dimension of the linear space

$$\mathfrak{G} = \{\alpha_1 + \alpha_2 \mid \alpha_1 \in \mathfrak{S}_{m-2}(S),\ \alpha_2 \in \mathfrak{S}_{m-2}(E),\ \alpha_{1\Gamma} = \alpha_{2\Gamma}\},$$

where $\Gamma = ES$. For an arbitrary element $A \in \mathfrak{S}_{m-2}(\mathfrak{M})$, we have

$A_S \in \mathfrak{H}_{m-2}(S)$, $A_E \in \mathfrak{H}_{m-2}(E)$ and $(A_S)_\Gamma = (A_E)_\Gamma$; thus $A \to A_S + A_E$ is a linear mapping of $\mathfrak{H}_{m-2}(\mathfrak{M})$ into $\mathfrak{H}$. It follows from the proposition (II) that, for every element $\alpha_1 + \alpha_2$ of $\mathfrak{H}$, there exists one and only one element A of $\mathfrak{H}_{m-2}(\mathfrak{M})$ with $A_E = \alpha_2$. This element A satisfies $(\alpha_1 - A_S)_\Gamma = \alpha_{2\Gamma} - (A_E)_\Gamma = 0$, while $\Gamma = ES$ is a hyperplane section of S. Hence, by the proposition (I), we get $A_S = \alpha_1$. Thus we see that $A \to A_S + A_E$ is a one-to-one mapping of $\mathfrak{H}_{m-2}(\mathfrak{M})$ *onto* $\mathfrak{H}$. Consequently we obtain $g_{m-2}(\mathfrak{M}) = g_{m-2}(S + E)$, q. e. d.

[1] In case there exists on $\mathfrak{M}$ at least one non-trivial meromorphic m-ple differential W_0, the *canonical divisor* K on $\mathfrak{M}$ may be defined by $K = (W_0)$. Then the dimension of $\mathfrak{W}(S; \mathfrak{M})$ is equal to the dimension of the *adjoint system* $|K + S|$ of S increased by 1.

[2] For the special case in which $\mathfrak{M}$ is a Kählerian *surface*, see K. Kodaira, "The Theorem of Riemann-Roch on Compact Analytic Surfaces," *Am. J. Math.*, **73**, 813–874 (1951), §6.

[3] See Kodaira, *loc. cit.*, §1, where the reader will find a summary of the results in the theory of harmonic integrals which are necessary for the present note.

[4] de Rham, G., and Kodaira, K., "Harmonic Integrals" (mimeographed notes), Princeton (1950), pp. 63–68; see also Kodaira, *loc. cit.*, p. 817.

[5] Kodaira, *loc. cit.*, Theorem 1.4.

[6] Kodaira, *loc. cit.*, Theorem 1.3.

[7] Hartogs, F., *Acta Math.*, **32**, 57–79 (1909).

[8] Lefschetz, S., *L'Analysis situs et la géométrie algébrique*, Paris, 1924, pp. 89–91.

[9] It can be shown that $a(V)$ is equal to the *arithmetic genus* of V. See K. Kodaira, "Arithmetic Genera of Algebraic Varieties," these PROCEEDINGS, **38**, 527–533 (1952).

ARITHMETIC GENERA OF ALGEBRAIC VARIETIES

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Communicated by Hermann Weyl, March 10, 1952

Let us consider an arbitrary irreducible non-singular algebraic variety V_n of dimension n imbedded in a projective space, and denote by $g_k(V_n)$ $(1 \leqq k \leqq n)$ the number of linearly independent k-ple differentials of the first kind attached to V_n. In the present short note, we shall prove a conjecture of F. Severi[1] to the effect that *the arithmetic genus* $p_a(V_n)$ *of* V_n *is represented in the form*

$$p_a(V_n) = g_n(V_n) - g_{n-1}(V_n) + g_{n-2}(V_n) - + \cdots\cdots + (-1)^{n-1} g_1(V_n). \tag{1}$$

It follows from (1) that *the arithmetic genus* $p_a(V_n)$ *is a birational invariant of* V_n, since each $g_k(V_n)$ is known to be a birational invariant[2] of V_n.

1. We set

$$a(V_n) = g_n(V_n) - g_{n-1}(V_n) + g_{n-2}(V_n) - + \cdots\cdots + (-1)^{n-1}g_1(V_n). \tag{2}$$

Then we have to prove the identity

$$p_a(V_n) = a(V_n). \tag{3}$$

First we consider an arbitrary but fixed irreducible non-singular variety $\mathfrak{M} = V_m$ of dimension $m \geqq 2$. A linear system $\{A\}$ of effective divisors A on $\mathfrak{M}$ without fixed components corresponds to a rational mapping[3] of $\mathfrak{M}$ into a projective space $\mathfrak{S}$, and this correspondence is one-to-one. *We shall say that the linear system $\{A\}$ is sufficiently ample if the corresponding rational mapping induces a biregular imbedding of $\mathfrak{M}$ into $\mathfrak{S}$.* By virtue of theorems of Bertini, *a general member A of a sufficiently ample linear system $\{A\}$ is an irreducible non-singular subvariety of $\mathfrak{M}$.* It is obvious that, *if $\{A\}$ is sufficiently ample, the complete linear system $|A|$ is also sufficiently ample.* Again, *if $\{A\}$ is sufficiently ample and if $\{B\}$ has neither fixed components nor base points, then the complete linear system $|A + B|$ is also sufficiently ample.* Assuming that $\mathfrak{M}$ is imbedded in a (fixed) projective space $\mathfrak{S}_d$ of dimension d, we denote by E_h the section of $\mathfrak{M}$ cut out by a general hypersurface $\mathfrak{F}_h$ of order h. Then it follows from the latter proposition that, *for a given divisor D, the complete linear system $|D + E_h|$ is sufficiently ample if h is sufficiently large.* In what follows we denote by S, T, U irreducible non-singular subvarieties of $\mathfrak{M}$ of dimension $m - 1$ and by K the *canonical divisor* on $\mathfrak{M}$. Incidentally, for an arbitrary divisor D on an irreducible non-singular variety V, we denote by $|D|_V$ the complete linear system *on* V consisting of all effective divisors which are linearly equivalent to D.

THEOREM 1.[4] *If S belongs to a sufficiently ample linear system on $\mathfrak{M}$, then we have*

$$\dim |K + S|_{\mathfrak{M}} = a(\mathfrak{M}) + a(S) - 1. \tag{4}$$

THEOREM 2.[5] *Assume that $m = \dim \mathfrak{M} \geqq 3$. If S belongs to a sufficiently ample linear system on $\mathfrak{M}$ and if the intersection $S \cdot T$ is an irreducible non-singular variety with the intersection multiplicity 1, then we have*

$$\dim |K + S + T|_{\mathfrak{M}} = a(\mathfrak{M}) + a(S) + a(T) + a(S \cdot T) - 1. \tag{5}$$

The above two theorems can be proved with the help of the theory of harmonic integrals.[6]

THEOREM 3. *If $|S|_{\mathfrak{M}} = |T|_{\mathfrak{M}}$, then $a(S) = a(T)$.*

Proof: In case $m(= \dim \mathfrak{M}) = 2$, the *genus* $a(S)$ of the *curve* S on the surface $\mathfrak{M}$ is determined by the relation[7] $2a(S) - 2 = I(S, K + S)$, where $I(S, K + S)$ denotes the intersection number of S and $K + S$. Hence $a(S) = a(T)$ if $|S|_{\mathfrak{M}} = |T|_{\mathfrak{M}}$. Now, assume that our theorem is

true for $m < n$ and consider the case in which $m = n > 2$. Let $E = E_1$ be a general hyperplane section of $\mathfrak{M}$. Then S or T cuts out on E an irreducible non-singular subvariety $S \cdot E$ or $T \cdot E$, respectively. Since $|K + S + E|_{\mathfrak{M}} = |K + T + E|_{\mathfrak{M}}$, we get, using (5), $a(S) + a(S \cdot E) = a(T) + a(T \cdot E)$. On the other hand, it follows from $|S|_{\mathfrak{M}} = |T|_{\mathfrak{M}}$ that $|S \cdot E|_E = |T \cdot E|_E$. Hence, by the induction hypothesis, we get $a(S \cdot E) = a(T \cdot E)$. Consequently we obtain $a(S) = a(T)$, q. e. d.

THEOREM 4. *Let U be an irreducible non-singular subvariety of $\mathfrak{M}$ of dimension $m - 1$. If $|S|_{\mathfrak{M}}$ is sufficiently ample, then $|K + S + U|_{\mathfrak{M}}$ cuts out on U a complete linear system.*

Proof: Denote by $|K + S + U|_{\mathfrak{M}} \cdot U$ the linear system cut out by $|K + S + U|_{\mathfrak{M}}$ on U and by K_U the canonical divisor on U. Clearly, a general member S of $|S|_{\mathfrak{M}}$ cuts out on U an irreducible non-singular variety $S \cdot U$ with the intersection-multiplicity 1, and $|S \cdot U|_U$ is sufficiently ample. Now, we infer readily that $|K + S + U|_{\mathfrak{M}} \cdot U$ is contained in the complete linear system $|K_U + S \cdot U|_U$, while

$$\dim \{|K + S + U|_{\mathfrak{M}} \cdot U\} = \dim |K + S + U|_{\mathfrak{M}} - \dim |K + S|_{\mathfrak{M}} - 1.$$

By virtue of (4) and (5), we get

$$\begin{aligned} \dim |K_U + S \cdot U|_U &= a(U) + a(S \cdot U) - 1, \\ \dim |K + S|_{\mathfrak{M}} &= a(\mathfrak{M}) + a(S) - 1, \\ \dim |K + S + U|_{\mathfrak{M}} &= a(\mathfrak{M}) + a(S) + a(U) + a(S \cdot U) - 1. \end{aligned}$$

Consequently, we see that $\dim \{|K + S + U|_{\mathfrak{M}} \cdot U\} = \dim |K_U + S \cdot U|_U$. This shows that $|K + S + U|_{\mathfrak{M}} \cdot U$ coincides with the complete linear system $|K_U + S \cdot U|_U$, q. e. d.

COROLLARY. *Let U be an irreducible non-singular subvariety of $\mathfrak{M}$ of dimension $m - 1$. Then, for an arbitrary divisor D on $\mathfrak{M}$, $|D + E_h|_{\mathfrak{M}}$ cuts out on U a complete linear system if h is sufficiently large.*

2. We denote by E^k the intersection of k general hyperplane sections of $\mathfrak{M}$. For each $k \leqq m - 1$, E^k is an irreducible non-singular subvariety of $\mathfrak{M}$ of dimension $m - k$ and, as one readily infers from Theorem 3, $a(E^k)$ is independent of the choice of E^k, i.e., $a(E^k)$ is determined uniquely by $\mathfrak{M}$, $|E|_{\mathfrak{M}}$ and k. E^m is a set consisting of a finite number of points on $\mathfrak{M}$. *We denote by $a(E^m)$ the number of points of E^m decreased by 1.* Clearly $a(E^m) + 1$ is equal to the *order* of $\mathfrak{M}$.

LEMMA 1. *Assume that the intersection $S \cdot T$ is an irreducible non-singular variety with the intersection multiplicity 1. If $|U|_{\mathfrak{M}} = |S + T|_{\mathfrak{M}}$ and if $|S|_{\mathfrak{M}}$ and $|U|_{\mathfrak{M}}$ are both sufficiently ample, then we have*

$$a(U) = a(S) + a(T) + a(S \cdot T). \tag{6}$$

Proof: The lemma is an immediate consequence of (4) and (5).

THEOREM 5. *We have*

$$a(E_h) + (-1)^{m-1} = \sum_{k=1}^{m} \binom{h}{k} [a(E^k) + (-1)^{m-k}]. \tag{7}$$

Proof: We prove the relation (7) by induction on m and h. In case $m = 2$, (7) is an immediate consequence of the relation $2a(E_h) - 2 = I(E_h, E_h + K)$. Now, assume that (7) is true for $m \leqq n - 1$ and consider the case in which $m = n > 2$. For $h = 1$, (7) is valid trivially. Assume therefore that (7) is true for E_{h-1}. Since $|E_h|_{\mathfrak{M}} = |E_{h-1} + E|_{\mathfrak{M}}$, we get, using (6),

$$a(E_h) = a(E_{h-1}) + a(E) + a(E_{h-1} \cdot E), \tag{8}$$

while, by the induction hypothesis,

$$a(E_{h-1}) + (-1)^{n-1} = \sum_{k=1}^{n} \binom{h-1}{k} [a(E^k) + (-1)^{n-k}],$$

$$a(E_{h-1} \cdot E) + (-1)^{n-2} = \sum_{k=1}^{n-1} \binom{h-1}{k} [a(E^{k+1}) + (-1)^{n-k-1}].$$

Inserting these two identities into (8), we obtain (7), q. e. d.

It follows from Theorem 5 that *there exist a polynomial* $v(l; \mathfrak{M})$ *in* l *of degree* m *and an integer* l_0 *(depending on* $\mathfrak{M}$*) such that*

$$\dim |E_l|_{\mathfrak{M}} = v(l; \mathfrak{M}) - 1, \qquad \text{for } l \geqq l_0. \tag{9}$$

In fact: choose a positive integer l_0 such that $|E_{l_0} - K|_{\mathfrak{M}}$ is sufficiently ample and set $l = h + l_0$. Then, denoting by S an irreducible non-singular variety belonging to $|E_{l_0} - K|_{\mathfrak{M}}$, we get, using (5),

$$\dim |E_l|_{\mathfrak{M}} = \dim |K + E_h + S|_{\mathfrak{M}} = a(\mathfrak{M}) + a(E_h) + a(S) + a(E_h S) - 1$$

while, as (7) shows, $a(E_h)$ or $a(E_h S)$ is a polynomial in h of degree m or $m - 1$, respectively. This shows the existence of a polynomial $v(l; \mathfrak{M})$ satisfying (9).

This polynomial $v(l; \mathfrak{M})$ is called the *postulation of* $\mathfrak{M}$. *The arithmetic genus*[8] $p_a(\mathfrak{M})$ *of* $\mathfrak{M}$ *is defined by*

$$p_a(\mathfrak{M}) = (-1)^m [v(0; \mathfrak{M}) - 1]. \tag{10}$$

LEMMA 2. *For sufficiently large* l*, we have*

$$v(l; S) = \dim |E_l|_{\mathfrak{M}} - \dim |E_l - S|_{\mathfrak{M}}. \tag{11}$$

Proof: For large l, we have $v(l; S) = \dim |E_l S|_S + 1$, while, since, as the Corollary to Theorem 4 shows, $|E_l|_{\mathfrak{M}}$ cuts out on S a complete linear system, $\dim |E_l S|_S$ is equal to $\dim |E_l|_{\mathfrak{M}} - \dim |E_l - S|_{\mathfrak{M}} - 1$ Hence we get (11).

From (11) we get immediately the following

THEOREM 6. *If* $|S|_{\mathfrak{M}} = |T|_{\mathfrak{M}}$, *then* $v(l;\ S) = v(l;\ T)$ *and* $p_a(S) = p_a(T)$.

THEOREM 7. *Assume that the intersection* $S \cdot T$ *is an irreducible non-singular variety with the intersection-multiplicity 1. If* $|U|_{\mathfrak{M}} = |S + T|_{\mathfrak{M}}$, *then we have*

$$v(l;\ U) = v(l;\ S) + v(l;\ T) - v(l;\ S \cdot T), \tag{12}$$

$$p_a(U) = p_a(S) + p_a(T) + p_a(S \cdot T) \tag{13}$$

Proof: Considering $S \cdot T$ as a divisor on T, we get, by (11), $v(l;\ S \cdot T) = \dim |E_l T|_T - \dim |E_l T - ST|_T$ for large l, while, by virtue of the Corollary to Theorem 4,

$$\dim |E_l T|_T = \dim |E_l|_{\mathfrak{M}} - \dim |E_l - T|_{\mathfrak{M}} - 1,$$

$$\dim |E_l T - ST|_T = \dim |E_l - S|_{\mathfrak{M}} - \dim |E_l - S - T|_{\mathfrak{M}} - 1$$

for sufficiently large l. Hence we obtain

$$v(l;\ S \cdot T) = \dim |E_l|_{\mathfrak{M}} - \dim |E_l - S|_{\mathfrak{M}} - \dim |E_l - T|_{\mathfrak{M}} + \dim |E_l - U|_{\mathfrak{M}}.$$

Using (11), we obtain (12) from this immediately. The formula (13) is an immediate consequence of (12).

3. Let $\mathfrak{S}_d$ be a projective space of dimension d and V_{d-1}, V_{d-2}, $\cdots$, V_{d-k}, $\cdots$ be irreducible non-singular subvarieties of $\mathfrak{S}_d$ such that each V_{d-k} is the section of V_{d-k+1} cut out by a hypersurface $\mathfrak{F}'_{n_k}$ of order n_k (in particular $V_{d-1} = \mathfrak{F}'_{n_1}$). Denoting by $\mathfrak{F}_l$ a general hypersurface of order l, we infer readily that the canonical divisor on V_{d-k} is given by $K_{d-k} = \mathfrak{F}_{s_k} \cdot V_{d-k}$, where $s_k = n_1 + n_2 + \cdots\cdots + n_k - d - 1$. Since $|\mathfrak{F}_l \cdot V_{d-k+1} - V_{d-k}|_{V_{d-k+1}} = |\mathfrak{F}_{l-n_k} \cdot V_{d-k+1}|_{V_{d-k+1}}$ and $v(l;\ V_{d-k+1}) = \dim |\mathfrak{F}_l \cdot V_{d-k+1}|_{V_{d-k+1}} + 1$ for large l, we get, by (11),

$$v(l;\ V_{d-k}) = v(l,\ V_{d-k+1}) - v(l - n_k;\ V_{d-k+1}). \tag{14}$$

It follows from the well-known formula $v(l;\ \mathfrak{S}_d) = \binom{l+d}{d}$ that $v(-l-d-1;\ \mathfrak{S}_d) = (-1)^d v(l;\ \mathfrak{S}_d)$. Starting from this, we get, with the help of (14), the formula

$$v(-l + s_k;\ V_{d-k}) = (-1)^{d-k} v(l;\ V_{d-k}) \tag{15}$$

successively for $k = 1, 2, 3, \ldots$. Now, using (4), we get, for sufficiently large l,

$$\begin{aligned} v(l + s_k;\ V_{d-k}) &= \dim |\mathfrak{F}_{l+s_k} \cdot V_{d-k}|_{V_{d-k}} + 1 \\ &= \dim |K_{d-k} + \mathfrak{F}_l \cdot V_{d-k}|_{V_{d-k}} + 1 \\ &= a(V_{d-k}) + a(\mathfrak{F}_l \cdot V_{d-k}), \end{aligned}$$

while, as (7) shows, $a(\mathfrak{E}_l \cdot V_{d-k})$ is a polynomial in l whose constant term equals $(-1)^{d-k}$. Hence, using (15), we obtain

$$a(V_{d-k}) + (-1)^{d-k} = v(s_k;\ V_{d-k}) = (-1)^{d-k} v(0;\ V_{d-k}).$$

This proves that $p_a(V_{d-k}) = a(V_{d-k})$ for the varieties $V_{d-k} = \mathfrak{E}'_{n_1} \cdot \mathfrak{E}'_{n_2} \cdot \ldots \cdot \mathfrak{E}'_{n_k}$ $(k = 1, 2, \ldots)$.

4. Now we shall prove the identity $p_a(V_n) = a(V_n)$ by induction on n. In the simplest case in which $n = 1$, this identity is an immediate consequence of the theorem of Riemann-Roch. We assume therefore that the identity $p_a(V_n) = a(V_n)$ is true for $n \leqq m - 1$ and consider an arbitrary but fixed irreducible non-singular variety V_m of dimension $m \geqq 2$ imbedded in a projective space $\mathfrak{S}_d$ of dimension $d > 2m + 1$. Take $d - m - 1$ general hypersurfaces $\mathfrak{E}_{n_1}, \mathfrak{E}_{n_2}, \ldots, \mathfrak{E}_{n_{d-m-1}}$ of sufficiently high orders $n_1, n_2, \ldots, n_{d-m-1}$ *containing* V_m and set $V_{d-k} = \mathfrak{E}_{n_1} \cdot \mathfrak{E}_{n_2} \cdot \ldots \cdot \mathfrak{E}_{n_k}$ $(k = 1, 2, \ldots, d - m - 1)$. Then, as one readily infers from the principle of counting constants and the theorems of Bertini, each V_{d-k} is an irreducible non-singular variety. Furthermore V_{d-k} is the section of V_{d-k+1} cut out by the hypersurface $\mathfrak{E}_{n_k}$. Hence, denoting by $b_\nu(V_{d-k})$ the ν-Betti number of V_{d-k}, we get, by a theorem of S. Lefschetz,[9]

$$b_\nu(V_{m+1}) = b_\nu(V_{m+2}) = \ldots = b_\nu(V_{d-1}) = b_\nu(\mathfrak{S}_d), \qquad \text{for } \nu \leqq m.$$

In particular $b_1(V_{m+1}) = b_1(\mathfrak{S}_d) = 0$, $b_2(V_{m+1}) = b_2(\mathfrak{S}_d) = 1$, and consequently $b_{2m}(V_{m+1}) = b_2(V_{m+1}) = 1$. Set $\mathfrak{M} = V_{m+1}$ and denote by E_h the section of $\mathfrak{M}$ cut out by a general hypersurface $\mathfrak{E}_h$ of order h. Again set $\delta(S) = p_a(S) - a(S)$ for an arbitrary irreducible non-singular subvariety S of $\mathfrak{M}$ of dimension m. Then, comparing (13) with (6), we get

$$\delta(U) = \delta(S) + \delta(T), \qquad \text{for } |U|_{\mathfrak{M}} = |S + T|_{\mathfrak{M}}, \qquad (16)$$

provided that $|S|_{\mathfrak{M}}$ and $|U|_{\mathfrak{M}}$ are both sufficiently ample and that $S \cdot T$ is an irreducible non-singular variety with the intersection-multiplicity 1, since $p_a(S \cdot T) = a(S \cdot T)$ by the induction hypothesis. Now, consider the subvariety V_m of $\mathfrak{M} = V_{m+1}$ as a $2m$-cycle on $\mathfrak{M}$. Since $b_{2m}(\mathfrak{M}) = 1$, we can find two positive integers q, r such that $qV_m \sim E_r$, where $\sim$ denotes the homology with respect to integral coefficients, while it follows from $b_1(\mathfrak{M}) = 0$ that[10] $qV_m \sim E_r$ implies $|qV_m|_{\mathfrak{M}} = |E_r|_{\mathfrak{M}}$. Choose h so large that $|kV_m + E_h|_{\mathfrak{M}}$ is sufficiently ample for $k = 0, 1, 2, \ldots, q$ and let S_k be a general member of $|kV_m + E_h|_{\mathfrak{M}}$. Then, since $|S_k|_{\mathfrak{M}} = |S_{k-1} + V_m|_{\mathfrak{M}}$, we get, by (16), $\delta(S_k) = \delta(S_{k-1}) + \delta(V_m)$, and therefore $\delta(S_q) = q \cdot \delta(V_m) + \delta(S_0)$, while $|S_q|_{\mathfrak{M}} = |E_{r+h}|_{\mathfrak{M}}$ and $|S_0|_{\mathfrak{M}} = |E_h|_{\mathfrak{M}}$. Consequently we obtain $q \cdot \delta(V_m) = \delta(E_{r+h}) - \delta(E_h)$. On the other hand, the result of Section 3 shows that $\delta(E_l) = 0$ for every $E_l = \mathfrak{E}_{n_1} \cdot \mathfrak{E}_{n_2} \cdot \ldots \cdot \mathfrak{E}_{n_{d-m-1}} \cdot \mathfrak{E}_l$. Hence we get $\delta(V_m) = 0$, proving that $p_a(V_m) = a(V_m)$, q. e. d.

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[4] Kodaira, K., "On the Theorem of Riemann-Roch for Adjoint Systems on Kählerian Varieties," these PROCEEDINGS, **38,** 522–527 (1952). Theorem 3.

[5] Kodaira, K., *loc. cit.*, Theorem 4.

[6] See Kodaira, K., *loc. cit.*

[7] See Kodaira, K., *Am. J. Math.*, **73,** 854 (1951).

[8] See Muhly, M. T., and Zariski, O., *Trans. Am. Math. Soc.*, **69,** 82 (1950).

[9] Lefschetz, S., *L'Analysis situs et la géométrie algebrique*, Paris, 1924, p. 89.

[10] de Rham, G., and Kodaira, K., "Harmonic Integrals" (mimeographed notes), Princeton (1950), p. 110, Theorem 4.

A GENERALIZATION OF A THEOREM OF HODGE

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Communicated by S. Lefschetz, April 30, 1952

Let M be a real Riemannian manifold of dimension n and of class C^∞. At an arbitrary point $p \in M$ choose an orthonormal basis $e_{(k)}$, $k = 1, 2, \ldots, n$, for the tangent vectors at p. Given any real numbers α^k, $k = 1, 2, \ldots, n$, satisfying

$$\sum_{k=1}^{n} (\alpha^k)^2 = 1,$$

the unit vector

$$\lambda = \sum_{k=1}^{n} \alpha^k e_{(k)}$$

determines a unique geodesic issuing from p. Let r denote distance from p along this geodesic, and introduce the normal coordinates $x^k = \alpha^k r$, $k = 1, 2, \ldots, n$. These coordinates are obviously determined up to an orthogonal transformation.

Let $S_\epsilon(p)$ denote the geodesic sphere of radius ϵ and center at the point p. We say that M is uniformly smooth if: (i) there is a positive number η such that any two points of M whose distance does not exceed η can be joined by a unique geodesic; (ii) throughout any $S_\epsilon(p)$ the partial derivatives up to the fourth order, with respect to the above coordinates at p, of the metric tensor are bounded by a number independent of p.

We assume that M is *uniformly smooth* and *metrically complete*. Then the following two theorems are true:

(A) Any differential form φ of degree q, $0 \leq q \leq n$, which has finite norm and satisfies $\Delta\varphi = 0$, also satisfies $d\varphi = \delta\varphi = 0$.

(B) Given any closed q-form α which has a finite norm, there exists a unique norm-finite harmonic form which has the same periods on all q-cycles with compact carriers.

Theorem B is a generalization of the classical theorem of Hodge. Let C be the group of closed q-forms φ having finite norm (cocycles), and let B be the subgroup of C composed of forms $\varphi = d\psi$ (coboundaries). The quotient group $H = C/B$ defines a cohomology group of dimension q which is isomorphic to the space of harmonic q-forms. If M is compact (closed), H is just the cohomology group of dimension q in the usual sense.

Theorems A and B are false if the hypothesis of metric completeness is dropped. A proof of A and B based on the heat-equation method introduced by Milgram and Rosenbloom in reference 1 will appear in the *Annals of Mathematics*.

[1] Milgram, A. N., and Rosenbloom, P. C., "Harmonic Forms and Heat Conduction I: Closed Riemannian Manifolds," PROC. NATL. ACAD. SCI., **37**, 180–184 (1951).

POWER-ASSOCIATIVE COMMUTATIVE ALGEBRAS OF DEGREE TWO

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Communicated by A. A. Albert, April 25, 1952

It has been shown that every central simple power-associative commutative algebra of degree greater than two and characteristic greater than two is a classical Jordan algebra. Such an algebra of degree one and characteristic zero is isomorphic to its center.[1] In this paper we shall show that a simple power-associative commutative algebra of degree two and characteristic $p > 5$ need not be a Jordan algebra. We do this by constructing an example.

Let $A = F[x]$ be an associative algebra over the field F with unity element e and let $x^0 = e$ and $x^p = 0$. Assume that e is the sum of two orthogonal idempotents, u and v. Then $A = uA + vA$. We take two quantities y_0 and y_1 and consider the system $S = A + uA + y_0A + y_1A$. Further assume that $a + ub + y_0c + y_1d = 0$, for a, b, c, d in A, if and only if $a = b = c = d = 0$. Since A has dimension p, the vector space S has dimension $4p$.

Define multiplication in S by assuming the commutative law, the product formulas

$$(y_i x^\beta)(x^\alpha u) = [(y_i x^\beta)x^\alpha]u = [(y_i x^\beta)u]x^\alpha = {}^1/_2(y_i x^\beta)x^\alpha = {}^1/_2 y_i x^{\alpha+\beta}, \quad (1)$$

and the formula

$$(y_i x^\alpha)(y_j x^\beta) = (i - j)(\alpha - \beta)x^{\alpha+\beta-1}, \quad (2)$$

where $i, j = 0, 1$ and $\alpha, \beta = 0, 1, \ldots, p - 1$. Relation (1) implies that e is the unity element of S. Note that $i = 1$, $\alpha = p$, $j = \beta = 0$ in (2) gives $(y_1 x^p)(y_0 x^0) = p x^{p-1}$. Since $x^p = 0$, $p x^{p-1} = 0$, and it follows that the characteristic of S must divide p. We assume that the characteristic of S is p.

When p is greater than 5, the power-associativity of S is equivalent to the identity[2]

$$4(xy)(zw) + 4(xz)(yw) + 4(xw)(yz) = x[(yz)w + (yw)z + (wz)y] + y[(xz)w + (xw)z + (wz)x] + z[(xy)w + (xw)y + (yw)x] + w[(xy)z + (xz)y + (yz)x]. \quad (3)$$

If the characteristic of S is 3 or 5 other identities must also be satisfied if S is to be power-associative. We shall content ourselves with showing that S is power-associative for $p > 5$. This is done by substituting all possible combinations of four elements in (3).

In order to facilitate the proof of the power-associativity of S we make some auxiliary computations. For convenience we write $ux = g$ and $vx = h$ so that $x = g + h$. Then (1) and (2) imply

$$[(y_i x^\alpha)g^\beta](y_j x^\gamma) + (y_i x^\alpha)[(y_j x^\gamma)g^\beta] = {}^1/_2(y_i x^{\alpha+\beta})(y_j x^\gamma) + {}^1/_2(y_i x^\alpha)(y_j x^{\beta+\gamma}) = {}^1/_2(i - j)(\alpha + \beta - \gamma)x^{\alpha+\beta+\gamma-1} + {}^1/_2(i - j)(\alpha - \beta - \gamma)x^{\alpha+\beta+\gamma-1} = (i - j)(\alpha - \gamma)x^{\alpha+\beta+\gamma-1}.$$

We write the result as

$$[(y_i x^\alpha)g^\beta](y_j x^\gamma) + (y_i x^\alpha)[(y_j x^\gamma)g^\beta] = [(y_i x^\alpha)(y_j x^\gamma)]x^\beta. \quad (4)$$

Another relation may be obtained by replacing g with h. Direct computation shows that

$$[(y_i x^\alpha)(y_j x^\beta)](y_j x^\gamma) + [(y_i x^\alpha)(y_j x^\gamma)](y_j x^\beta) = (i - j)(2\alpha - \beta - \gamma)y_j x^{\alpha+\beta+\gamma-1}, \quad (5)$$

and

$$[(y_i x^\alpha)g^\beta]g^\gamma + [(y_i x^\alpha)g^\gamma]g^\beta + (y_i x^\alpha)(g^\beta g^\gamma) = y_i x^{\alpha+\beta+\gamma}. \quad (6)$$

First consider the case where one of the variables in (3) has the form $y_i x^\alpha$. If the other variables are powers of x then we use the fact that $(y_i x^\alpha)x^\beta = y_i x^{\alpha+\beta}$ to see that (3) is satisfied. If a power of g or h is used

in place of a power of x then (1) implies that each term of (3) is halved so we have disposed of this case.

Let us proceed to substitutions involving exactly two variables of the form $y_i x^\alpha$. The subscripts of these two elements must differ if we are to obtain non-zero products. Because of linearity and of the orthogonality of g and h we need only consider the two sets of values $x = y_1x^\alpha$, $y = y_0x^\beta$, $z = g^\gamma$, $w = h^\delta$, and $x = y_1x^\alpha$, $y = y_0x^\beta$, $z = g^\gamma$, $w = g^\delta$. After making the first of these substitutions and using (4) we need to show that

$$4[(y_1x^\alpha)g^\gamma][(y_0x^\beta)h^\delta] + 4[(y_1x^\alpha)h^\delta][(y_0x^\beta)g^\gamma] = (y_1x^\alpha)([(y_0x^\beta)g^\gamma]h^\delta + [(y_0x^\beta)h^\delta]g^\gamma) + (y_0x^\beta)([(y_1x^\alpha)g^\gamma]h^\delta + [(y_1x^\alpha)h^\delta]g^\gamma) + ([(y_1x^\alpha)(y_0x^\beta)]x^\delta)g^\gamma + ([(y_1x^\alpha)(y_0x^\beta)]x^\gamma)h^\delta.$$

We use the multiplication table of S to reduce this equality to

$$(y_1x^{\alpha+\gamma})(y_0x^{\beta+\delta}) + (y_1x^{\alpha+\delta})(y_0x^{\beta+\gamma}) = {}^1/_2(y_1x^\alpha)(y_0x^{\beta+\gamma+\delta}) + {}^1/_2(y_0x^\beta)(y_1x^{\alpha+\gamma+\delta}) + [(y_1x^\alpha)(y_0x^\beta)]x^{\gamma+\delta}.$$

Then (2) implies that this expression holds in S. Relation (3) with values $x = y_1x^\alpha$, $y = y_0x^\beta$, $z = g^\gamma$, $w = g^\delta$ gives

$$4[(y_1x^\alpha)(y_0x^\beta)](g^\gamma g^\delta) + 4[(y_1x^\alpha)g^\gamma][(y_0x^\beta)g^\delta] + 4[(y_1x^\alpha)g^\delta][(y_0x^\beta)g^\gamma] = (y_1x^\alpha)(y_0x^{\beta+\gamma+\delta}) + (y_0x^\beta)(y_1x^{\alpha+\gamma+\delta}) + [(y_1x^\alpha)(y_0x^\beta)]g^{\gamma+\delta}$$

where we have used (4) and (6). This simplifies to

$$4(\alpha - \beta)g^{\alpha+\beta+\gamma+\delta-1} + (\alpha + \gamma - \beta - \delta)x^{\alpha+\beta+\gamma+\delta-1} + (\alpha + \delta - \beta - \gamma)x^{\alpha+\beta+\gamma+\delta-1} = (\alpha - \beta - \gamma - \delta)x^{\alpha+\beta+\gamma+\delta-1} + (-\beta + \alpha + \gamma + \delta)x^{\alpha+\beta+\gamma+\delta-1} + 4(\alpha - \beta)g^{\alpha+\beta+\gamma+\delta-1}$$

which is satisfied identically.

If three of the variables of (3) have the form y_ix^α, we need only make the substitution $x = y_1x^\alpha$, $y = y_0x^\beta$, $z = y_0x^\gamma$, $w = g^\delta$. We apply (4) and (5) after making the substitution. Then (3) is equivalent to

$$4[(y_1x^\alpha)(y_0x^\beta)][(y_0x^\gamma)g^\delta] + 4[(y_1x^\alpha)(y_0x^\gamma)][(y_0x^\beta)g^\delta] = (y_0x^\beta)([(y_1x^\alpha)(y_0x^\gamma)](x^\delta + g^\delta)) + (y_0x^\gamma)([(y_1x^\alpha)(y_0x^\beta)](x^\delta + g^\delta)) + g^\alpha[(2\alpha - \beta - \gamma)y_0x^{\alpha+\beta+\gamma-}].$$

Every summand of this expression is a multiple of $y_0x^{\alpha+\beta+\gamma+\delta-1}$ and computation of the coefficients gives $2(\alpha - \beta) + 2(\alpha - \gamma) = {}^3/_2(\alpha - \gamma) + {}^3/_2(\alpha - \beta) + {}^1/_2(2\alpha - \beta - \gamma)$ as desired.

There remains the case with all four variables of (3) having the form y_ix^α. The only set of values of this type giving non-zero products is

$x = y_1x^\alpha$, $y = y_1x^\beta$, $z = y_0x^\gamma$, $w = y_0x^\delta$. We then wish to show that

$$4[y_1x^\alpha)(y_0x^\gamma)][(y_1x^\beta)(y_0x^\delta)] + 4[(y_1x^\alpha)(y_0x^\delta)][(y_1x^\beta)(y_0x^\gamma)] =$$
$$(y_1x^\alpha)[(2\beta - \gamma - \delta)y_0x^{\beta+\gamma+\delta-1}] + (y_1x^\beta)[(2\alpha - \gamma - \delta)y_0x^{\alpha+\gamma+\delta-1}] +$$
$$(y_0x^\gamma)[(\alpha + \beta - 2\delta)y_1x^{\alpha+\beta+\delta-1}] + (y_0x^\delta)[(\alpha + \beta - 2\gamma)y_1x^{\alpha+\beta+\gamma-1}].$$

Each term is a multiple of $x^{\alpha+\beta+\gamma+\delta-2}$ and a check of the coefficients shows that (3) is satisfied. Thus the algebra S is power-associative for any characteristic $p > 5$.

The algebra S is simple. For suppose B is a non-zero ideal of S. If $u = g^0$ is in S then $2(y_1x)g^0 = y_1x$ and $(y_1x)(y_0x^0) = x^0 = e$ is in S. Therefore B is S. Similarly, if $v = h^0$ is in B, $B = S$. If x^α is in B, it follows that $[(y_ix^0)x^\alpha]y_0 = \alpha x^{\alpha-1}$ is in B. After a finite number of applications of this argument we have $x^0 = e$ in B. In the same manner we may show that g^α in B implies $B = S$. Whenever y_ix^α is in B then $(y_ix^\alpha)(y_jx^0) = (i - j)\alpha x^{\alpha-1}$ is in B. We take $j \neq i$ and then have $x^{\alpha-1}$ in B. Then B cannot be a proper ideal.

In the decomposition of S relative to the idempotent u, $S_u(1) = F[g] = uA$, $S_u(0) = F[h] = vA$, and $S_u(1/2) = y_1A + y_0A$. It follows from (1) that S is u-stable; that is, $S_u(1/2)S_u(\lambda) \subseteq S_u(1/2)$ for $\lambda = 0, 1$. However, S is not stable. We prove this statement by showing that the subalgebra $T = A + y_0A + y_1A$ is not stable. The subalgebra T has order $3p$ and is obviously simple. Consider the element $w = y_1x + {}^1/_2y_0x^0$ in T. Then $w^2 = (y_1x)(y_0x^0) = x^0 = e$ and $[(e + w)/2]^2 = (e + w)/2$. That is, $z = {}^1/_2(e + y_1x + {}^1/_2y_0x^0)$ is an idempotent. Also, $e - z = (e - w)/2$ is idempotent, $e = (e - z) + z$ and $(e - z)z = z - z^2 = 0$ so T has degree 2. We have $(y_1x^0)z = {}^1/_2y_1x^0$ and $(zx)z = {}^1/_2(x + y_1x^2 + {}^1/_2y_0x)z = {}^1/_4(x + y_1x^2 + {}^1/_2y_0x + y_1x^2 + {}^1/_2y_0x + x) = zx$. Therefore y_1x^0 is in $T_z(1/_2)$ and $zx = {}^1/_2(x + y_1x^2 + {}^1/_2y_0x)$ is in $T_z(1)$. We compute $(y_1x^0)(zx) = {}^1/_2(y_1x - {}^1/_2e)$ and $[(y_1x^0)(zx)]z = {}^1/_4(y_1x - {}^1/_2e - {}^1/_2y_1x + {}^1/_2e - {}^1/_4y_0x^0) = {}^1/_4({}^1/_2y_1x - {}^1/_4y_0x^0)$. Since $[(y_1x^0)(zx)]z \neq {}^1/_2[(y_1x^0)(zx)]$, T, and therefore S, is not stable. Every Jordan algebra is stable so S and T are not Jordan algebras.[3]

† This work was carried out in part with the aid of the Office of Naval Research.

[1] These results are given in Albert, A. A., *Trans. Amer. Math. Soc.*, **69**, 503–527 (1950) with the restriction that the characteristic be different from 2, 3, and 5. The results were extended for characteristic 3 and 5 in the author's unpublished Ph.D. dissertation.

[2] See Albert, A. A., *Summa Brasiliensis Mathematicae*, **2**, 21–33 (1948).

[3] The subalgebra T was originally constructed independently of S and in a much more complicated form. The author is indebted to A. A. Albert for noticing that T is a subalgebra of S and that S is not stable although u-stable.

SOME RESULTS ON ABELIAN GROUPS[1]

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Communicated by A. A. Albert, April 25, 1952

1. Introduction.—The author is engaged in the preparation of a monograph on infinite abelian groups. Besides exposition of known theory, it is planned to include some new material. Since publication will presumably be somewhat delayed, it seems appropriate to make this brief announcement of the new results.

2. Definitions.—Let G be an abelian group. A subgroup of G is *fully invariant* (*characteristic*) if it admits every endomorphism (automorphism) of G. A subgroup H of G is *divisible* if $nH = H$ for all n. G is *reduced* if its only divisible subgroup is 0. For most questions it suffices to consider reduced groups. G is *primary* (for the prime p) if every element has order a power of p. For a reduced primary group G we define a transfinite series as follows: $G_{\alpha+1} = pG_\alpha$ for every ordinal α, and G_α is the intersection of the preceding subgroups if α is a limit ordinal. If a nonzero element x lies in G_α but not in $G_{\alpha+1}$, we write $h(x) = \alpha$. We write $h(0) = \infty$, with the agreement that ∞ exceeds any ordinal. Let P be the subgroup of elements of order p. The dimension of $(P \cap G_\alpha)/(P \cap G_{\alpha+1})$, regarded as a vector space over the integers mod p, is the αth *Ulm invariant* of G, written $f(\alpha)$. If G is countable, it is characterized by its Ulm invariants.

3. The Ring of Endomorphisms.—THEOREM 1. *Let G and H be primary abelian groups, E and F their rings of endomorphisms. Then any isomorphism of E and F is induced by a group isomorphism of G and H. In particular, every automorphism of E is inner.*

This theorem was proved by Baer,[2] for groups of bounded order having at least three linearly independent elements of maximum order, with the aid of his theory of the lattice of subgroups. The author's proof uses ring-theoretic techniques, reminiscent of the classical treatment of automorphisms of matrix rings.

4. Characteristic Subgroups.—In Theorems 2 and 3 we confine ourselves to a reduced primary group G, but the theorems are easily extended to the non-reduced case. Let α_i be a monotone increasing sequence of ordinals; it is allowed to be ∞ from some point on. The set of x satisfying $h(p^i x) \geqq \alpha_i$ for all i is manifestly a fully invariant subgroup of G.[3]

THEOREM 2. *Let G be a reduced primary group such that every two elements can be embedded in a countable direct summand (this is in particular true if G itself is countable, or if it has no elements of infinite height). Then every fully invariant subgroup of G has the form just described. The lattice of*

fully invariant subgroups is distributive and even satisfies the infinite distributive law $A \cap (\cup B_i) = \cup (A \cap B_i)$; *the dual infinite law is, however, satisfied if and only if G has no elements of infinite height.*

Characteristic subgroups are fully invariant, with an exceptional case generalizing that discovered by Shoda[4] and Baer.[5]

THEOREM 3. *The hypothesis on G is the same as in Theorem 2. Then the following two statements are equivalent: (a) G has a characteristic subgroup which is not fully invariant, (b)* $p = 2$, *and there exist ordinals* α, β *with* $\beta > \alpha + 1$ *such that the Ulm invariants* $f(\alpha)$ *and* $f(\beta)$ *are both 1.*

5. Complete Modules.—In this final section we are concerned with modules over the p-adic integers (any complete discrete valuation ring would do in place of the p-adic integers). If M is such a module, we say it has no elements of infinite height if the intersection of $p^i M$ is 0. These submodules may then be taken as neighborhoods of 0 for a topology, and we say that M is *complete* if it is complete in this topology.

THEOREM 4. *A complete module over the p-adic integers is the completion of a direct sum of cyclic modules. The cardinal numbers giving the number of cyclic summands of each order are a complete set of invariants.*[6]

From Theorem 4 it is possible to deduce the answer to a question raised by the author in reference 7.

THEOREM 5. *Any module M over the p-adic integers has a direct summand of rank one. In particular, M is indecomposable if and only if it has rank one.*

Theorem 4 can also be applied to the problem of determining which abelian groups can be compact. Let G be a compact abelian group. It is known that its component C of the identity is its maximal divisible subgroup. Hence C is (algebraically, not topologically!) a direct summand of G. It is also known that the totally disconnected group G/C is the complete direct product of groups H_p, each of which is a complete module over the p-adic integers. Thus G belongs to a class of abelian groups which is completely characterized by a set of cardinal numbers. The author has not fully solved the problem of determining which sets of cardinals are eligible to be those of a compact group, but this appears to be of lesser interest. Indeed it would seem to be a good idea to call a group G "algebraically compact" if it has the form $G = C \oplus H$, where C is divisible and H is a complete direct sum of complete modules over the p-adic integers. Then at any rate negative theorems carry greater force when they assert that a group is not algebraically compact. Sample theorem: an abelian group with an infinite cyclic direct summand is not algebraically compact.

[1] This work was supported in part by the Office of Naval Research.

[2] *Ann. Math.*, **44**, 192–227 (1943).

[3] This description of fully invariant subgroups is a convenient modification of that of Shiffman, *Duke Math. J.*, **6**, 579–597 (1940).

[4] *Math. Zeit.*, **31**, 611–624 (1930).

[5] *Proc. London Math. Soc.*, **39**, 481–514 (1935).

[6] The torsion-free case of Theorem 4 was discovered by Isidore B. Fleischer and appears in his dissertation.

[7] *Trans. Am. Math. Soc.*, **72**, 327–340 (1952).

NATURE OF INHERITED RESISTANCE TO VIRUSES AFFECTING THE NERVOUS SYSTEM

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Read before the Academy, April 28, 1952

Introduction.—The vast majority of viruses which attack the nervous system of human beings and animals produce recognizable disease and death in only a small proportion of infected individuals. A variety of factors pertaining to the virus, the host, or both, may influence the outcome of infection. The studies to be presented here were designed to elucidate the intricate biological mechanisms which form the basis of this important phenomenon.

Previous studies by other investigators have established that plants and animals may possess an inherited resistance to various infectious and noxious agents, including viruses. The studies of Lynch and Hughes[1] with the virus of yellow fever and those of Webster[2] with the viruses of louping ill and St. Louis encephalitis provided the first experimental evidence that the genetic constitution of the host can determine the outcome of mammalian viral infections. A most important contribution to this subject was Webster's demonstration that the inherent resistance or susceptibility of mice to the virus of St. Louis encephalitis was correlated with the level of viral multiplication in the brain, not only in the intact animal[3] but also in simple cultures containing the minced brain tissue.[4] Earlier attempts to establish the manner in which such resistance is inherited by animals, and to segregate the factors which are genetically affected, were complicated by the fact that no uniformly resistant animals were available.

Experimental Results.—In 1944, the author accidentally discovered that the albino mice which had been bred for over 25 years at the Rockefeller Institute at Princeton, N. J., were 100 per cent resistant to the 17 D strain of yellow fever virus that is widely used for human vaccination. Swiss mice, intracerebrally inoculated with this virus, invariably die after exhibiting

paralysis and other signs of involvement of the nervous system. In the Princeton Rockefeller Institute mice (henceforth called PRI), intracerebral inoculation of the 17 D virus, in the largest as well as the smallest doses, results in multiplication at a level only $^{1}/_{10,000}$ to $^{1}/_{100,000}$ of that achieved in Swiss mice, and in the development of specific antibodies—but all mice survive without showing any clinical signs of disease. The possibility was considered that a latent virus in the PRI mice might interfere with the multiplication of the yellow fever virus, but none was found in tests with brain, viscera, intestines and feces. This circumstance presented an unusual opportunity for genetic analysis by the classical methods of

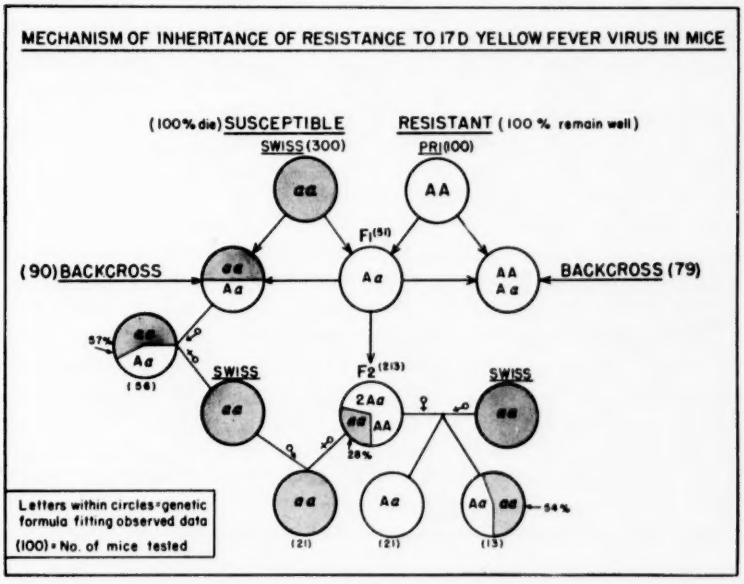

FIGURE 1

interbreeding. Such studies were begun in 1950, and this communication is a progress report of the results obtained thus far.

Figure 1 presents a diagrammatic summary of the tests with the 17 D virus. The sex of the parents and progeny is not indicated because no evidence of sex-linkage was found. Since all F_1 progeny survived without signs of disease, it is clear that resistance to this virus is inherited as a dominant. The approximate 3:1 ratio of resistants to susceptibles in the F_2 progeny, the exact 1:1 ratio in the backcross of F_1 to susceptible Swiss mice, and the 100 per cent resistance in the backcross of F_1 to resistant PRI mice, are all in accord with the Mendelian laws for a single pair of

autosomal genes. However, in 1934, Sewall Wright[5] demonstrated that such ratios are not critical criteria for the operation of a single pair of genes and that the critical experiment must test the genetic nature of the backcross and F_2 progeny by breeding them with the recessive stock. This was done, as shown in figure 1, and the results obtained were again compatible with a single factor hypothesis. It is realized, however, that it is difficult to rule out other possibilities involving multiple genes, which do not assort

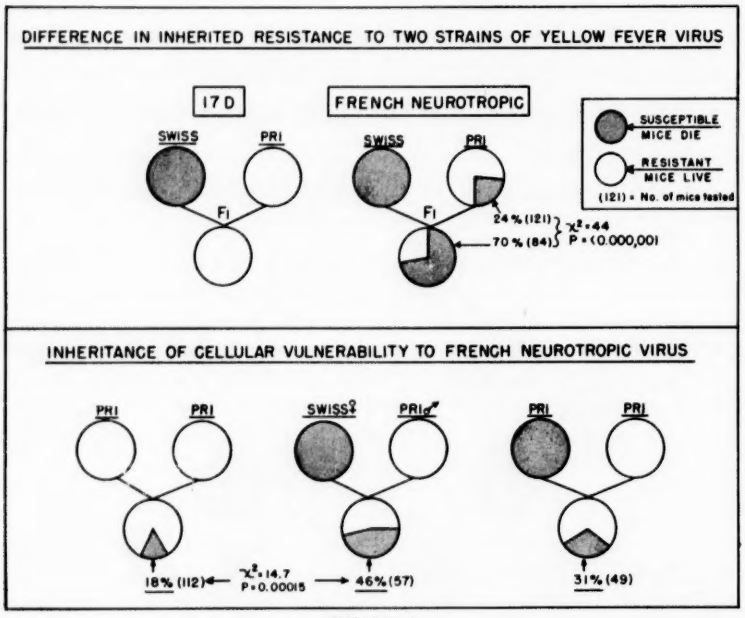

FIGURE 2

This figure shows the mortality resulting from tests on parental and F_1 groups of mice. The genetic aspects are discussed in the text.

at random in the gametes but are linked with only a limited amount of crossing over. Thus, if resistance to 17 D were dependent on two pairs of dominant, linked genes, with 10 per cent crossing over, the backcross and F_2 progeny and the progeny resulting from crossing these with the recessive stock would be modified so slightly from the single factor expectancy that only exceptionally large numbers of animals might yield results of sufficient statistical significance to permit a decision. However, whatever the exact number of genes may be, it is clear that the resistance of PRI mice to the 17 D virus depends on a factor, which depresses viral multiplication, and

is inherited as an "all or none" character, because the progeny which segregated out of the various crossings were phenotypically like the parent resistant or susceptible stock, i.e., the resistant mice showed no signs of disease and the susceptible ones exhibited the same incubation period and high level of viral multiplication as the Swiss mice.

All this held good as long as the tests were restricted to mature animals. When newborn, 1-day or 2-day old PRI mice were inoculated with 17 D virus, all mice died after an incubation period of 10 days or longer. PRI mice inoculated at 3 to 5 days of age behaved irregularly, and beyond 5 days of age, the suckling mice behaved like the mature animals. Tests for viral multiplication in the succumbing suckling PRI mice showed that the peak levels were as low as in adult PRI mice which remained well. It was apparent, therefore, that at low levels of viral multiplication the tissues of the newborn mice were more vulnerable than those of older mice.

The concept of cellular vulnerability again came under consideration when it was found that the "French neurotropic" strain of yellow fever virus killed a certain proportion of adult PRI mice. This strain of virus has had over 250 serial passages in the brains of Swiss mice, and although its level of multiplication is not significantly higher than that of the 17 D strain, it kills mice more rapidly. The results summarized in figure 2 show that 24 per cent of adult PRI mice died after intracerebral inoculation of the "French neurotropic" virus. The virus multiplied at the same low level in the PRI mice that died as in those that remained well, and both of these levels were about the same as the peak levels of the 17 D strain. There was no evidence that the virus particles which grew out in the brains of the dying mice were different from those in the brains of mice which remained well, since passage to new PRI mice yielded a mortality of only 20 per cent. It appeared, therefore, that some of the PRI mice were more vulnerable than others at similar low levels of viral multiplication, and the question arose as to whether cellular vulnerability might also be genetically determined.

Since the multiplication-depressing factor is inherited as a dominant, it is possible to compare the cellular vulnerability of the F_1 progeny (Swiss $\times$ PRI) mice with that of PRI mice, without having it complicated by the factor of viral multiplication. In repeated tests, in which approximately one million infective doses of "French neurotropic" virus were inoculated intracerebrally in groups of 20 to 30 mature PRI mice, the mortality ranged from 20 to 30 per cent, with an average of 24 per cent. Crosses made between PRI and the highly inbred CFW, Swiss mice produced 84 F_1 animals, which exhibited an average mortality of 70 per cent after intracerebral inoculation of the same large dose of "French neurotropic" virus. In litters of 7 or more mice the mortality was never less than 50 per cent, and in 2 families of 8 mice, each resulting from a cross

of PRI females with Swiss males, the mortality was 100 per cent. In general, it made no difference whether the mother was of PRI or Swiss stock. This result, which is highly significant statistically, indicates that the character of the parents influences the behavior of the progeny. When F_1 mice were backcrossed to PRI mice (not shown in figure 2), 60 per cent of the progeny (12/20) died following inoculation of the French neurotropic virus. These results strongly suggest that cellular vulnerability to the "French neurotropic" strain of yellow fever is also genetically determined. The fact that PRI parents, which were proved to be resistant, i.e., possessed a low cellular vulnerability, yielded 18 per cent of highly vulnerable progeny suggests not only that high cellular

TABLE 1

SELECTIVE ACTION OF MULTIPLICATION INHIBITION FACTOR OF PRI MICE ON VARIOUS VIRUSES PROLIFERATING IN MOUSE BRAIN

ACTION	VIRUS
Multiplication inhibited. PRI mice completely or partly resistant	Yellow Fever
	Dengue Fever
	West Nile Fever
	Japanese B Encephalitis
	St. Louis Encephalitis
	Russian Spring-Summer Encephalitis
Multiplication not affected. PRI mice fully susceptible	Western Equine Encephalitis
	Eastern Equine Encephalitis
	Venezuelan Equine Encephalitis
	Poliomyelitis
	Mouse Encephalomyelitis—"TO"
	Rabies
	Lymphocytic Choriomeningitis
	Herpes Simplex
	Vesicular Stomatitis
	Rift Valley Fever

vulnerability might involve a recessive gene or genes but also that many of the unselected PRI stock are heterozygous for this character. The F_1 progeny resulting from crossing resistant PRI mice with Swiss mice yielded 46 per cent of highly vulnerable animals, which again would suggest that the Swiss mice are probably homozygous for the recessive genes, responsible for high vulnerability, and that the resistant (low vulnerable) PRI mice are predominantly heterozygous. The progeny resulting from crossing susceptible PRI with resistant PRI contained a lower percentage (31 per cent) of highly vulnerable individuals than in the tests in which Swiss mice were used; the difference between this result and the 18 per cent of vulnerable progeny produced by resistant PRI parents is in the right direction but not large enough for statistical significance. A great deal more work will have to be done on segregating the unselected PRI mice

by individual matings with Swiss mice before further genetic analysis of the factor of cellular vulnerability will become possible. While the viral-multiplication factor yielded "all or none" results, there is an indication that cellular vulnerability is not an "all or none" character, since a small proportion of mice develop weakness or extensive paralysis of the extremities but do not die.

The genetic factor in PRI mice, which depresses multiplication of the yellow fever virus, was found to exert a similar effect on the viruses of dengue fever, West Nile fever, Japanese B, St. Louis and Russian spring-summer encephalitis, but was without effect on a large group of other viruses listed in table 1. It is of interest to note that the viruses, which are affected by this genetic factor, are also linked together by a chain of common antigens,[6] not shared by the other viruses.

TABLE 2

OCCURRENCE OF VARIANT STRAINS OF VIRUS CAPABLE OF OVERCOMING INHERITED RESISTANCE OF HOST

VIRUS	STRAIN	NO. OF PASSAGES IN SWISS MICE	BEHAVIOR IN PRI MICE		PEAK MULTIPLICATION IN BRAINS OF SWISS MICE
			MORTALITY, %	PEAK MULTIPLICATION IN BRAIN	
Japanese B encephalitis	Korea	4	13	2.4, 3.0, 3.8[a]	8.0[a]
	Nakayama	43	50	3.3	7.8
	Nakayama	±70	100	7.6	9.5
St. Louis encephalitis	Winkler	8	6	3.6	7.8
	Webster No. 3	"Hundreds" over 17 years	90	4.2	9.0

[a] Reciprocal of log of LD_{50}.

Only the dengue viruses behaved like the 17 D yellow fever virus in that all the PRI mice were uniformly resistant regardless of dosage. The behavior of the West Nile, Japanese B and St. Louis viruses varied markedly with the strain of virus and the number of passages it had had in Swiss mice prior to test in PRI mice. Some of the strains killed no PRI mice, others a varying proportion, and still others killed all the animals. Table 2 presents some of the data which indicate that viruses may overcome the inherited resistance of the host in at least 2 different ways: (1) by possessing or developing variants which can kill at low levels of viral multiplication, and (2) by the development of variants which are no longer inhibited by the inherited multiplication-depressing factor. We cannot be certain that these variations in the populations of particles belonging to any one virus may not occur in nature, and, the genetically resistant mouse can be used to investigate this possibility. For example, it was found that 2 of the 3 strains of West Nile virus recovered in Dr. Paul's laboratory from Egyptian sera,[7] killed only a small proportion of PRI mice, while the

third strain (Egypt 101), in the second mouse passage, killed all PRI mice although it multiplied at a level that was only $^1/_{4000}$ of that achieved in the brains of Swiss mice.

Summary and Discussion.—These studies, in brief, have shown that an analysis of inherited resistance to viruses must take into consideration not only factors which control viral multiplication but also the phenomenon of cellular vulnerability. The available data suggest that cellular vulnerability also may be a genetically controlled character. With some strains of virus only one of these factors comes into play, with others, both. The demonstration that any one virus does not consist of a homogeneous population of particles, and that certain strains can possess or develop the capacity to overcome one or the other, or both, of these host barriers, poses a problem of considerable interest with regard to the potential periodic emergence of viruses with special epidemic or epizootic properties. These data provide a model which indicates how the viruses responsible for encephalitis and poliomyelitis may behave in populations of mixed or highly selected genetic constitution. The data also provide an explanation for the occurrence in some of the highly endemic areas of Africa of population groups which are almost uniformly resistant to yellow fever. The mixed occurrence of severe and mild infections among the South American Indians is in accord with the hypothesis that yellow fever was imported there from Africa after the discovery of America, and that not enough time has elapsed for the virus to kill off a significant number of those carrying the recessive genes for susceptibility. Finally, these studies provide us with a model of how nature prevents viruses from causing disease by allowing them to multiply just enough to produce immunity, but not enough to damage the tissues—a model that we may well investigate in great detail because an understanding of the underlying mechanisms would provide us with the ideal procedures for the chemoprophylaxis of virus diseases.

Acknowledgments.—The author wishes to thank Dr. Carl TenBroeck, director of the former Rockefeller Institute for Animal and Plant Pathology at Princeton, N. J., for sending the mice to start the PRI colony, and Mr. John H. Wallace for technical assistance.

[1] Lynch, C. J., and Hughes, T. P., *Genetics*, **21**, 104 (1936).
[2] Webster, L. T., *J. Exp. Med.*, **65**, 261 (1937).
[3] Webster, L. T., and Clow, A. D., *Ibid.*, **63**, 827 (1936).
[4] Webster, L. T., and Johnson, M. S., *Ibid.*, **74**, 489 (1941).
[5] Wright, S., *Genetics*, **19**, 537 (1934).
[6] Sabin, A. B., *Federation Proc.*, **8**, 410 (1949).
[7] Melnick, J. L., Paul, J. R., Riordan, J. T., Barnett, V. H., Goldblum, N., and Zabif, E., *Proc. Soc. Exp. Biol. Med.*, **77**, 661 (1951).

NATIONAL ACADEMY OF SCIENCES CONFERENCE ON QUANTUM-MECHANICAL METHODS IN VALENCE THEORY

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Communicated by R. S. Mulliken, April 3, 1952

"The underlying physical laws necessary for the mathematical theory of a large part of physics and the whole of chemistry are thus completely known, and the difficulty is only that the exact application of these laws leads to equations much too complicated to be soluble."—Dirac*

In April of 1950, at the Detroit ACS meeting, a group of chemists and physicists fell into a discussion of the inadequacies of valence theory, and what might be done to improve it. The need felt most acutely was a reliable table of the difficult integrals which inevitably turn up in valence calculations. Everyone agreed that a conference on such integrals would be desirable, both to find out which integrals would be most useful, and to avoid duplication of effort in computing them. Actually, the evaluation of integrals is not the only difficulty in quantum-mechanical valance theory, and as the idea of a conference developed, the range of topics was broadened. R. S. Mulliken of the University of Chicago took this idea and carried out the considerable work of organizing a conference. In this he was aided by the advice and cooperation of D. A. MacInnes, Chairman of the National Academy of Sciences Committee on Scientific Conferences. And so, with the sponsorship of the National Academy of Sciences and the financial support and encouragement of the Office of Naval Research, a Conference on Quantum-Mechanical Methods in Valence Theory was held on September 8 to 10, 1951, at Ram's Head Inn, Shelter Island, Long Island, with Mulliken as Conference Chairman.

Invitations were issued to a number of chemists and physicists, both in the United States and overseas. The following attended:

M. P. Barnett (Rayon Research Association, London)
T. H. Berlin (Johns Hopkins)
B. L. Crawford, Jr. (Minnesota)
C. A. Coulson (King's College, London)
H. Eyring (Utah)
J. O. Hirschfelder (Wisconsin)
G. E. Kimball (Columbia)
M. Kotani (Tokyo)
J. E. Lennard-Jones (Cambridge)
P. O. Löwdin (Uppsala)
D. A. MacInnes (Rockefeller Institute, New York)
H. Margenau (Yale)
J. E. Mayer (Chicago)
W. Moffitt (British Rubber Producers' Research Association, Herts)
R. S. Mulliken (Chicago)
R. G. Parr (Carnegie Institute of Technology)

K. S. Pitzer (California)
C. C. J. Roothaan (Chicago)
K. Rüdenberg (Chicago)
H. Shull (Iowa State)
J. C. Slater (Massachusetts Institute of Technology)
L. E. Sutton (Oxford)
C. W. Ufford (Pennsylvania)
J. H. Van Vleck (Harvard)
G. W. Wheland (Chicago)

The salubrious influence of the setting and accommodations engendered a corresponding freedom and informality in the discussions. These included reports on work completed, summaries of work in progress, indications of work contemplated, explanations of general viewpoints and arguments concerning them, helpful hints to quantum calculators, and concerted discussion and planning, particularly in the matter of integral evaluation. The diverse items, as collected in the ONR Report on the Conference,[1] give a picture of valence theory as it stands today.

The contributions to the discussions and the items which have been collected in the ONR Report seem to fall into six groups. The *first group* contained essentially straightforward applications of present-day theory. Most of these dealt with molecular problems; the atom was, however, represented by Ufford's calculations on d^6 and d^5s configurations of Mn II and Fe III, with configuration interaction. The molecules discussed ranged from hydrogen to pyrene, in order of increasing complexity of molecule and increasing empiricism of approach. Kotani is studying the polarizability of hydrogen, using an accurate wave function of the James-Coolidge type; his group is also treating acetylene as a pi-electron system by the LCAO molecular orbital method including configuration interaction. The nitrogen molecule is being studied in detail by several groups, including those of Herzfeld and Kopineck.

Shull presented a paper on the charge distribution in the carbon-oxygen bond of carbon dioxide which triggered an active discussion.[2] Several workers are now engaged in accurate calculations of dipole moments; useful comparisons can be made between theoretical and observed values of dipole moments and their dependence on internuclear distance, but the calculation must be detailed and the comparison critical.

There was little discussion at the Conference on aromatic molecules, which probably reflects a certain satisfaction with present pi-electron theory. Shull reported that varying the shielding constant in the Goeppert-Mayer-Sklar treatment of benzene does not alter the order of the computed electronic levels. Coulson described some calculations on the twisting of diphenyl which led to the conclusion that the equilibrium position might be as much as twenty degrees away from the planar. Wheland discussed magnetic anisotropy in a number of aromatic molecules, as computed by an empirical LCAO method, including overlap.[3]

The *second group* of contributions considered the quantum interpretation of chemical valence concepts—electron pairs, bond energies, bond orders,

hybridization, and chemical reactivity. Lennard-Jones indicated how the motion of electrons in molecules can be interpreted in terms of electron pairs whose positions are correlated, and showed that a useful approach to molecular problems can be based on "correlation functions." Berlin pointed out that the space around the nuclei in a diatomic molecule may be precisely divided into binding and antibinding regions, presence of an electron in the binding region tending to pull the nuclei together, in the antibinding region tending to make the nuclei fly apart. Mulliken disclosed the latest improvements in his "magic formula" for the semiempirical calculation of bond energies. He emphasized that pi-bonding contributions to the bond energies of multiple bonds are probably much larger than generally believed, and that isovalent hybridization (hybridization without change in valence) and non-bonded inner shell exchange repulsions are often important factors in determining molecular stability. Coulson and Mulliken discussed the concept of bond order, particularly with respect to the difficulties which arise when one attempts to define bond order for heteronuclear bonds. The concepts of bond order and hybridization both run into complications due to configuration interaction, as was pointed out by Daudel for the first and Mulliken for the second.

Coulson displayed maps of electron density in benzene, based in part on a Fermi-Thomas treatment of this molecule; these showed that the pi-electrons, on which so much attention has been focused, are actually "totally immersed in a sea of sigma-electrons."[4] This overwhelming density of sigma-electrons must produce overwhelming effects, and the question remains why the pi-electron theory of aromatic behavior is so successful.

Eyring discussed the possibility of calculating by quantum mechanics the structures of activated complexes, pointing out that this offered one of the greatest rewards of a successful valence theory. With the greater reward, the problem carries far greater difficulties, arising chiefly from the dearth of *a priori* knowledge of the equilibrium configuration of the activated complex. Even the simplest complex, H_3, has not yet been adequately treated.

The *third group* of contributions dealt with forces between molecules and between non-bonded atoms. First there was a review by Hirschfelder, which came to the unhappy conclusion that van der Waals' forces are still better determined from empirical data on gases than from theory. He stressed that a new attack on intermolecular forces, using new values of the appropriate integrals, would be in order, beginning with the simplest case of the inert gases. Next Margenau told of some recent work, including a treatment of the London forces between helium atoms which leads to a convergent rather than a divergent (asymptotic) series for the potential energy as a function of interatomic distance and which gives excellent agreement with experiment.

Very long range forces, operating over distances of 50 Å. or more, have recently been postulated by some biological workers. Coulson described an investigation of a possible origin of such forces, namely dispersion forces between extended oscillators. Deviations from the usual inverse sixth-order law were found, but not sufficiently startling ones. The model was crude, and the work is being extended; the conclusions are tentative. A specific type of intermolecular force, distinct from dispersion forces, and which he calls charge-transfer forces, was discussed by Mulliken, who attributes to these the formation of loose molecular complexes, and the interaction of electron donors and acceptors in general.

Van Vleck discussed a new source of information on intermolecular forces, line-breadth in microwave spectra. Line breadths furnish a measure of dipole–dipole interactions in the self-broadening of ammonia and hydrogen cyanide; of dipole-quadrupole interactions in the broadening of ammonia by carbon dioxide; and of dipole-quadrupole and even quadrupole-quadrupole interactions in the broadening of ammonia by nitrogen and oxygen.

The *fourth group* of contributions was concerned with the problem usually spoken of as that of configuration interaction or correlation energy. In attacking molecular problems the valence-bond or molecular orbital approaches have usually been used. Serious error is inherent in the first approximation of either approach; the error in the energy is known as "correlation energy" in the case of the molecular orbital approximation for the normal state of most molecules. In either approach, the wave function is built up from orbitals (one-electron wave functions), which may be localized or non-localized, atomic or nonatomic. No one has yet exhibited a quantitatively satisfactory first approximation. In principle, having chosen any approach and enough orbitals of any appropriate type, one can always converge to the correct answer by successive approximations. In practice, the convergence must be rapid for the method to be useful. For a given problem, this depends critically on the approach chosen and the starting orbitals used.

This choice of starting approach and starting wave functions is a core problem of quantum-mechanical valence theory. Much attention at the Conference was focused on this question. Slater discussed it not only in connection with magnetic studies, but also in the general case. Some of those present discussed the relative importance of correction terms added to conventional first approximations; these included Parr on helium,[5] Mulliken[6] and Lennard-Jones[7] on hydrogen, and Kotani on water and the methyl radical. Others discussed the choice of best first approximations of the orbital type. Kotani and Eyring each told of investigating new orbitals for hydrogen, and Eyring proposed the use of a new sort of "semi-localized orbital" which he found to work well for both the hydrogen and hydrogen fluoride molecules.[8]

Moffitt, after an incisive critique of the usual approaches, outlined a method to avoid the pitfalls of the orbital approximations. Rather than computing total molecular energies relative to separated nuclei and electrons, he computes directly the chemical dissociation energies, that is, molecular energies relative to the separated atoms. Instead of starting from orbitals, he starts from total atomic wave functions; these are of course not known, but by an elegant method, he obtains the molecular energies in terms of atomic term values and relatively small perturbation terms. Since the perturbation terms are small, they may be calculated from approximate wave functions, such as orbital functions. The needed atomic term values may be obtained either from separate atomic calculations, or (a more likely source) from spectroscopic observations on atoms. In principle, use of enough atomic term values ensures convergence; these must include ionic as well as neutral term values. The question which can only be settled by calculation is whether so many term values will be required that the method will be unwieldy. Some reported first results on oxygen and ethylene are encouraging.

A *fifth group* of contributions dealt with mathematical developments. Mayer outlined a formulation of the Hartree-Fock equations employing the charge density matrix and Löwdin reviewed numerical integration schemes for these equations. Slater and Ufford reported that IBM machines are now in use for the calculation of atomic self-consistent fields, but Slater cautioned that direct attack on problems of electronic structure is not yet within the reach of automatic computing machines; the machines can do much complex arithmetic, but the problem of the molecule has not yet been reduced to an arithmetic level.

The *sixth group* of contributions dealt explicitly with integrals. Inclusion of formerly neglected "small" integrals in molecular calculations of recent years has led to the realization that "completely theoretical" (as contrasted to "empirical") valence-theory calculations are capable of giving useful results, but with this realization comes an increased need for comprehensive table of integrals—in any valid molecular calculation all integrals must be included, unto the fifth center and the farthest neighbor. If one uses an approach based on atomic orbitals, in any problem one is ultimately led to integrals involving these orbitals alone. The integrals are of several types, which may be thought of as *generalizations* of the integrals met in the classical Heitler-London theory of the hydrogen molecule. Anyone who has worked through that theory, including the evaluation of the exchange integral by Sugiura, will appreciate the task now confronting the quantum mechanic. Small wonder that much discussion at the Conference centered on the topic of integrals, and that much interest was aroused by the possibility of cooperative work on the job ahead!

Groups now working on integrals include those at Cambridge University,

where Boys is systematically setting up to compute two-, three-, and four-center integrals using automatic computing machinery; the Max-Planck Institute in Göttingen, where Kopineck has completed a detailed compilation of known values for homopolar two-quantum two-center integrals and is extending the available values with new computations; Catholic University, where the program for evaluation of two-center integrals initiated by Sklar is being continued with the aid of IBM machines; University of Chicago, where Roothaan and Rüdenberg, with the cooperation of Shull and IBM of Iowa State, have well underway a program for computing and tabulating two-center integrals by explicit integration; King's College London, where Coulson and Barnett have laid out and begun computations with a comprehensive program making use of a scheme they have developed whereby a given integral (two-, three-, or four-center) may be calculated from a basic set of unit molecular integrals which they propose to tabulate; University of Tokyo, where Kotani is systematically extending his already well-known tables of two-center integrals; and University of Uppsala, where Löwdin and Lundquist have demonstrated the feasibility of evaluation by direct numerical integration of each and every integral encountered in a given problem.

More exploratory work needs to be done before a best method for computing molecular integrals, if one exists, will be apparent. Probably one method will turn out to be best for one type of integral, another best for another type. The Conference allowed the several groups of workers computing integrals to become acquainted with each other's work. As a consequence, efforts will likely no longer be expended on fruitless forcing of a method in a situation to which it is not well suited. The Conference also took a step which should greatly help those needing integrals in the immediate future: an informal Integrals Committee was established centered at the University of Chicago to collect and dispense integral information.

There is one question that Boys, Kimball and others have recently raised: Are Slater atomic orbitals really the best orbitals to use for molecular calculations? More attention might well be given this question before large-scale integral calculations are carried very far; the answer will have important bearing on the whole future course of quantum-mechanical methods in valence theory.

Dirac's statement quoted at the beginning of this summary has long been the hope and the despair of valence theoreticians: hopeful because the problem is solved "in principle"; despairing because the equations are "much too complicated to be soluble." The encouraging note of the Conference is that a frontal attack is at last being made on these equations—a hopeful, rather than a desperate, attack.

* Dirac, P. A. M., *Proc. Roy. Soc.*, **A123**, 714 (1929).

[1] The ONR Report on the Conference consists of numerous informal summaries and discussion remarks presented at the Conference, plus some longer papers, a total of some fifty-three items. A few of these have appeared in these PROCEEDINGS, **38,** 149–166, 364–378, 400–408 (1952). Copies of the ONR Report may be obtained, as long as the supply lasts, from L. M. McKenzie, Head, Physics Branch, Office of Naval Research, Department of the Navy, Washington 25, D. C.

[2] Shull, H., these PROCEEDINGS, **38,** 400–408 (1952).

[3] Wheland, G. W., and Matlow, H. L., *Ibid.*, **38,** 364–371 (1952).

[4] Coulson, C. A., March, N. H., and Altmann, S., *Ibid.*, **38,** 372–378 (1952).

[5] Taylor, G. R., and Parr, R. G., *Ibid.*, **38,** 154–160 (1952).

[6] Mulliken, R. S., *Ibid.*, **38,** 160–166 (1952).

[7] Lennard-Jones, J. E., *Ibid.*, **38,** 496–502 (1952).

[8] Mueller, C. P., and Eyring, H., *Ibid.*, **38,** 149–154 (1952).

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